STUDENT'S COLUMN

Duration and Convexity

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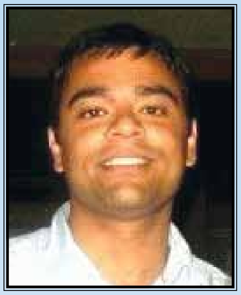


The price of a fixed income security negatively relates to the interest rate i.e. its yield. An upward movement in interest rate will cause a downward movement in price and vice versa. Interest rate risk is one of the prime concerns for a fixed income dealer. For measuring interest rate risk in fixed income bonds 2 approaches are used: (i) the full valuation approach (i.e. finding value by discounting the full stream of cashflows) and (ii) the duration and convexity approach. The concept of duration - convexity finds wide application in contemporary bond portfolio management. Primarily it is used to:

- Quantify the impact of changes of interest rates on bond prices.
- Asset – liability management of the banks, insurance companies, mutual funds etc.

Here we attempt to analyze the second approach.

There are 3 schools of thought on how to interpret duration:

- Duration is the first derivative of the price-yield function.
- Duration is the discounted mean term of cash flows, weighted by the present value of the bond.
- Duration is a measure of approximate sensitivity of the bond's value to change in yield.
- Let us analyze these three interpretations one by one.

[I] Duration as the first derivative of the price –yield function:

As per this interpretation price of a bond is a function of its yield i.e. $P_0 = f(y_0)$. Where P0 is the initial price of the bond and y0 is the initial yield of the bond. Curve of price - yield function is negatively sloped meaning as yield increases price decreases and when yield decreases price increases. Brook Taylor, an English mathematician in 1915 came out with the following equation to predict the fixed interest bond's value when its yield is changed and the relationship between yield and bond price is non-linear:

$P_1 = P_0 + (f'(y_0) * (y_1 - y_0)) + (2!^{-1} * f''(y_0) * (y_1 - y_0)^2) + (3!^{-1} * f'''(y_0) * (y_1 - y_0)^3) + - - - - -$

Where P_1 is the revised price of the bond due to change in the yield, y_1 is the revised yield and $f'(y_0)$ is the first derivative of the price - yield function and $f''(y_0)$ is the second derivative of the price - yield function and so on.

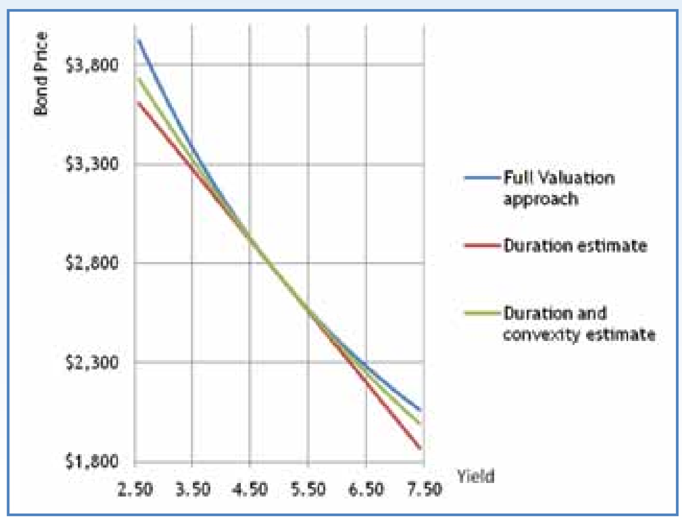
In case of fixed interest bonds, the first derivative is used for calculation of duration and the second derivative is used for calculation of convexity. Above result by Taylor can also be applied for approximating change in the value of a derivative contract. For example: Option on any underlying asset:
Revised price of the option =
Current price of the option + (f'(current price of the underlying asset) * change in the price of the underlying asset) + ($2!^{-1}$* f ''(current price of the underlying asset) * change in the price of the underlying asset^2) + - - - - - -
Here the first derivative is called delta and the second derivative is called gamma.

Coming back to bonds, Dollar duration for bonds is the negative of the first derivative of the price/yield function i.e. Dollar Duration = $-(dP_0 / dy_0) = -\Delta P/\Delta y$ (approximately). It is the first derivative of the price – yield function so it measures the slope of the tangent to the price – yield curve. Modified Duration = Dollar duration / Initial Price of the bond including any accrued interest at time 0.

1 basis point sensitivity = Dollar duration * .0001 → roughly approximates the change in bond price if yield changes by 0.01%.

This interpretation of duration can be used for analyzing change in price due to change in yield for a Zero Coupon Bond where price – yield curve is simple and differentiable.
$P_0 = CF_t / (1+y_t)^t$ => Dollar Duration = $-t * CF^t / (1+y_t)^{t+1} = (-t * P_0) / (1+y_t)$
And Modified Duration = $t / (1+y_t)$. CF_t is the redeemable amount at time t.

Duration measure assumes linear relationship between price and yield. In reality, the relationship between the changes in price and yield is convex. In the figure below, the curved line represents the change in prices when yield changes using the full valuation approach. The straight line, tangent to the curve, represents the estimated change in price calculated using the duration measure. The area between these two lines is the difference between the duration estimate and the actual price movement. As indicated, the larger the change in interest rates, the larger the error in estimating the price change of the bond.



Here we introduce convexity. Convexity is the second derivative of the price – yield function. Convexity is an approximation of the change in the bond's duration resulting from interest rate change. It is a measure of the curvature of the changes in the price of a bond in relation to changes in interest rates and is used to address the above approximation error. One may wonder that why we do not use the third or higher order term in the Taylor expansion series. Well, additional complexity introduced by the third or higher order terms do not contribute much to the improvement in the fit to the price yield curve.

Convexity * P_0 = d^2P / dy^2
For a Zero Coupon Bond, convexity
= $(d^2P / dy^2) / P_0$
= $((t+1) * t * P_0 / (1+y)^2) / P_0$
= $(t+1) * t / (1+y)^2$

Note:			
Compounding →	Annual	Semi - Annual	For conversion into years
Duration measure	Years	Half Years	Duration measure is divided by 2
Convexity measure	Years	Half Years squared	Convexity measure is divided by 4

Hence, the approximation for estimating bond's value because of change in yield can also be written as follows:
Revised Price
= Initial Price - (Modified duration * Initial price * $(y_1 - y_0)$)
+ ($2!^{-1}$ * Convexity * P_0 * $(y_1 - y_0)^2$)

(i) Actual price when yield changes (using full valuation approach) = Present value of the cashflows using changed yield.
(ii) Duration Estimate of actual price when yield changes (Ignoring second and higher order terms)
= P_0 - (Modified duration * P_0 * $(y_1 - y_0)$)
(iii) Duration and Convexity Estimate of Market price when yield changes (Ignoring third and higher order terms)
= P_0 - (Modified duration * P_0 * $(y_1 - y_0)$) + (0.5 * convexity * P_0*$(y_0 - y_1)^2$)

Example:

Zero Coupon Bond of 15 years redeemable at $100 - yield 8% with semiannual compounding			
Full Valuation Approach			
	Y_0	Y_0 + 0.01	Y_0 - 0.01
Yield p.a.	8%	9%	7%
Yield half year	4%	4.50%	3.50%
Market Price	$30.8319 = 100 / (1.04^(15*2))	$26.7 = 100 / (1.045^(15*2))	$35.6278 = 100 / (1.035^(15*2))
Duration and Convexity Calculations			
Duration Estimate of Market Price		$ 26.3850 = $ 30.8319 - $ 4.4469	$ 35.2788 = $ 30.8319 + $ 4.4469
Duration and Convexity Estimate of Market Price		$ 26.7163 = $ 30.8319 - $ 4.4469 + $ 0.3314	$ 35.6102 = $ 30.8319 + $ 4.4469 + $ 0.3314
Workings:			
Modified Duration (in half years)	28.85 =(15*2)/(1.04)	Modified Duration (in full years, ignoring negative sign)	14.42 =28.85/2
Dollar Duration (in half years)	889.38	Dollar Duration (in full years)	444.69 =889.38/2
Convexity (in half years)	859.84 =31*30/(1.04^2)	Convexity (in years)	214.96 =859.84/4
Duration adjustment to initial price when yield increases / decreases	4.4469 =(30*30.8319*0.01) / (2*1.04)	Convexity adjustment to initial price when yield increases / decreases	0.3314

Duration and Convexity Estimate of Market price is closer to the actual price using the full valuation approach as compared to the duration only estimate of the price.

Criticism of this interpretation:
This is an alternate approach for calculation of approximate change in the value of a Zero Coupon Bond because of change in yield. However, this is practically less useful interpretation as it does not contribute much in understanding the interest rate risk of a bond.

[II] Duration as a measure of cashflows weighted by the present value of the bond:
For a coupon paying bond price – yield function is not continuous. Many of the cashflows occur before actual maturity and the relative timing of these cashflows will affect the pricing of the bond. In order to deal with this Frederick Macaulay came up with the effective maturity of a bond as a measure of duration. Duration got defined as the weighted average of present value of cashflows based on the timing of the cashflows.
Macaulay's duration or Discounted Mean Term

$$= \frac{\sum_{k=1}^{n} k * CF(k) * (1+y_0)^{-k}}{\sum_{k=1}^{n} CF(k) * (1+y_0)^{-k}}$$

This represents the average time to wait for all cashflows to occur.
Macaulay's duration for Zero Coupon Bond = Term of that Zero Coupon Bond.
Note: Cap for duration of any fixed interest bond is given by (1+yeild) / yield.
Convexity

$$= \frac{\sum_{k=1}^{n} k * (k+1) * CF(k) * (1+y_0)^{-(k+2)}}{\sum_{k=1}^{n} CF(k) * (1+y_0)^{-k}}$$

Criticism of this interpretation:
(i) If a bond has duration of say 5 years, then it is not useful to think this measure in terms of time, but rather the bond has the price sensitivity to yield changes of a 5 year Zero Coupon bond.
(ii) For an interest only security say a Mortgage backed security duration

is negative. And it is difficult to interpret time as a negative quantity.

(iii) Certain collateralized mortgage obligation (CMO) bond classes have a greater duration than the underlying mortgage loans. That is, a CMO bond class can have duration of 50 although the underlying mortgage loans from which the CMO is created can have a maturity of 40 years.

[III] Duration as a measure of approximate sensitivity of bond's value to change in yield:

This is the interpretation which is commonly applied in practice. Duration is the approximate percentage change in the value for a 100 basis point change in rates.

Effective Duration = (P- - P+) / (2 * P_0 * Δ_y) ---- Equation 1

P_- = Price of bond with yield lesser than initial yield

P_+ = Price of bond with yield more than initial yield

P_0 = Price of bond with initial yield

Δ_y = (Initial Yield – Reduced Yield) or (Increased Yield – Initial Yield). This means that increased yield and reduced yield are equidistant from initial yield but with opposite signs.

Interpretation:

For duration of, say 5 this means that for 100 basis points change in yield, approximate change in price of bond will be 5%. That means if yield increases by 100 bps then there will be reduction in the bond price by 5% and vice versa.

Approximate percentage price change = - duration * Δ_y * 100 ---- Equation 2

Note:

- Δy in Equation 2 = Initial yield – changed yield for which bond's value is to be calculated.
- Duration estimate of revised price will always be lower than the revised value derived from the full valuation approach. Reason: Duration estimate is a tangent to the price – yield curve that always lie below the price – yield curve.

Relation between Macaulay's duration and Modified duration:

Modified Duration =

Macaulay's duration / (1 + (YTM / Number of times compounding in a year))

Some observations:

- Cēterīs paribus, the longer the maturity, greater will be the modified duration.
- For coupon paying bond, lower the coupon, generally the greater the modified and Macaulay duration of the bond.
- If the modified duration is greater, the price volatility will be greater.
- Cēterīs paribus, the higher the yield level, the lower the price volatility.

Convexity = (P+ + P- - 2*P_0) / (2 * P0 * Δy^2)

Convexity measure in isolation is not useful. It is the convexity adjustment that is required to be recognized along with duration's estimate of approximate price change.

Convexity Adjustment = Convexity * Δy^2 * 100 = Approximate percentage price increment to be added to approximate percentage price change computed using effective duration.

Convexity adjustment is always added in approximate price change calculated by duration except for the bonds with embedded options where it is to be deducted from the approximate price change calculated by duration.

How to compute duration of equity?

Duration of equity = Dividend yield / (1+ Dividend yield)

How to compute duration of perpetual bond?

Duration of perpetual bond = Bond yield / (1+ Bond yield)

Portfolio's duration:

Portfolio's duration is the weighted average duration: weighted on the market value of the bond constituting the portfolio. Portfolio duration of x means that for a 100 basis point change in the yield of all the constituent bonds in the portfolio, the market value of the portfolio will be approximately changed by x%. Here, it is implicitly assumed that the correlation between the yield change in each of the portfolio's bond of any maturity is 1.

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