Modelling the Zero Coupon Yield Curve:

A regression based approach

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Section 1: Introduction

What is the zero coupon yield curve?

Its importance in actuarial valuation

The Nelson Siegel term structure model for interest rates

What is the zero coupon yield curve?

- ZCB's are simple non-coupon bearing bonds
- A coupon-bearing bond can be 'stripped down' to a portfolio of ZCB's by considering each coupon as a separate ZCB
- The ZCYC is the relation between yield-to-maturity and maturity for such bonds
- Generally derived for each currency from the prevalent Government bond prices
- traditionally regarded as an important indicator of overall market conditions

Its importance in actuarial valuation

- Term structure of risk- free rates
- Liability valuation- ZCYC provides the appropriate discounting factors for valuing liabilities
- Asset returns - for risk-neutral valuation, expected returns from different asset classes are calibrated to risk-free rates derived from the ZCYC
- Aspects of yield curve that are most relevant to actuarial valuation models are –

  - It should give prices and yields close to the market

  - Its shape should capture market dynamics

The Nelson Siegel term structure model

Nelson and Siegel proposed the forward rate curve

$$r(m) = \beta_0 + \beta_1 * \exp(- m/\tau) + \beta_2 * (m/\tau) * \exp(-m/\tau) \quad (1)$$

This implies the yield curve –

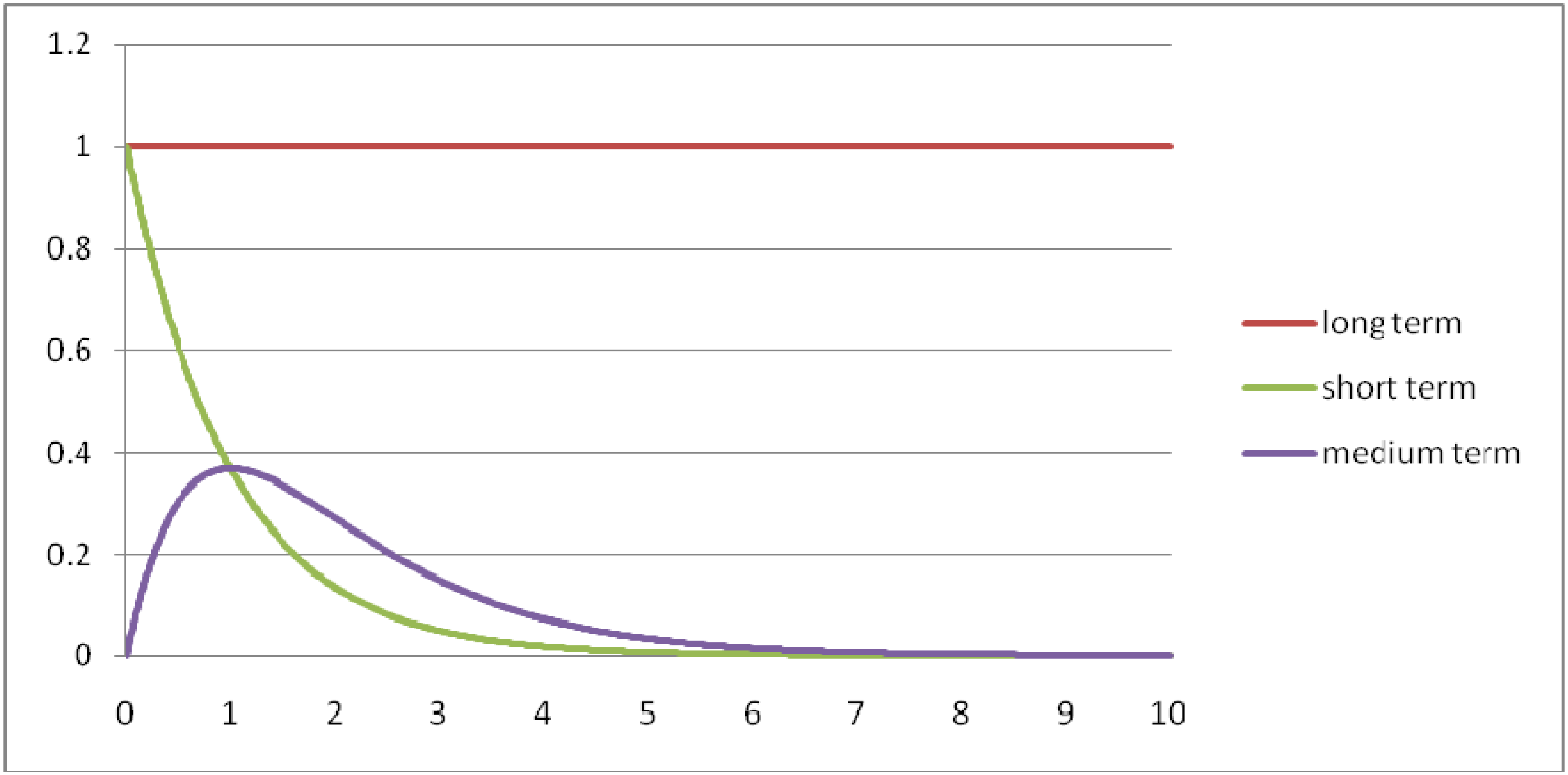
$$R(m) = \beta_0 + (\beta_1 + \beta_2) * (1 - \exp(-m/\tau))/(m/\tau) - \beta_2 * \exp(- m/\tau) \quad (2)$$

- **Popular for of its ease of interpretation and its parsimony**
- **The limiting value of R(m) as m gets large is β_0 and as m gets small is ($\beta_0 + \beta_1$), which are necessarily the same as for the forward rate function since r(m) is just an averaging of R(.)**

Interpretation of model parameters

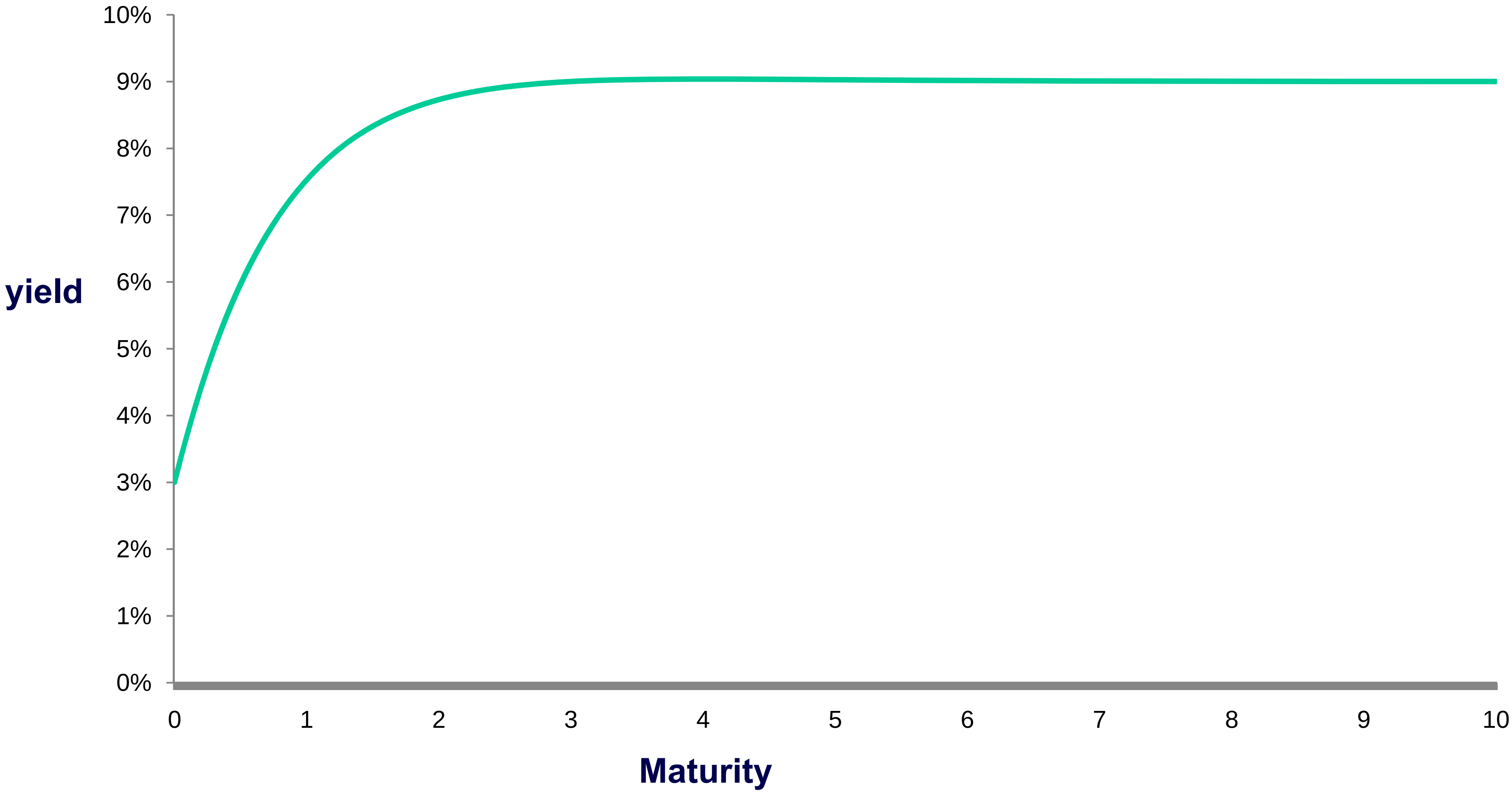
- T: governs the exponential decay rate; small values of τ produce fast decay and therefore fits curvature at short maturities, while large values of τ produce slow decay and gives better fit at long maturities
- β_0, β_1, β_2 : measure the strengths of the short-, medium-, and long-term components of the curve

Interpretation of model parameters



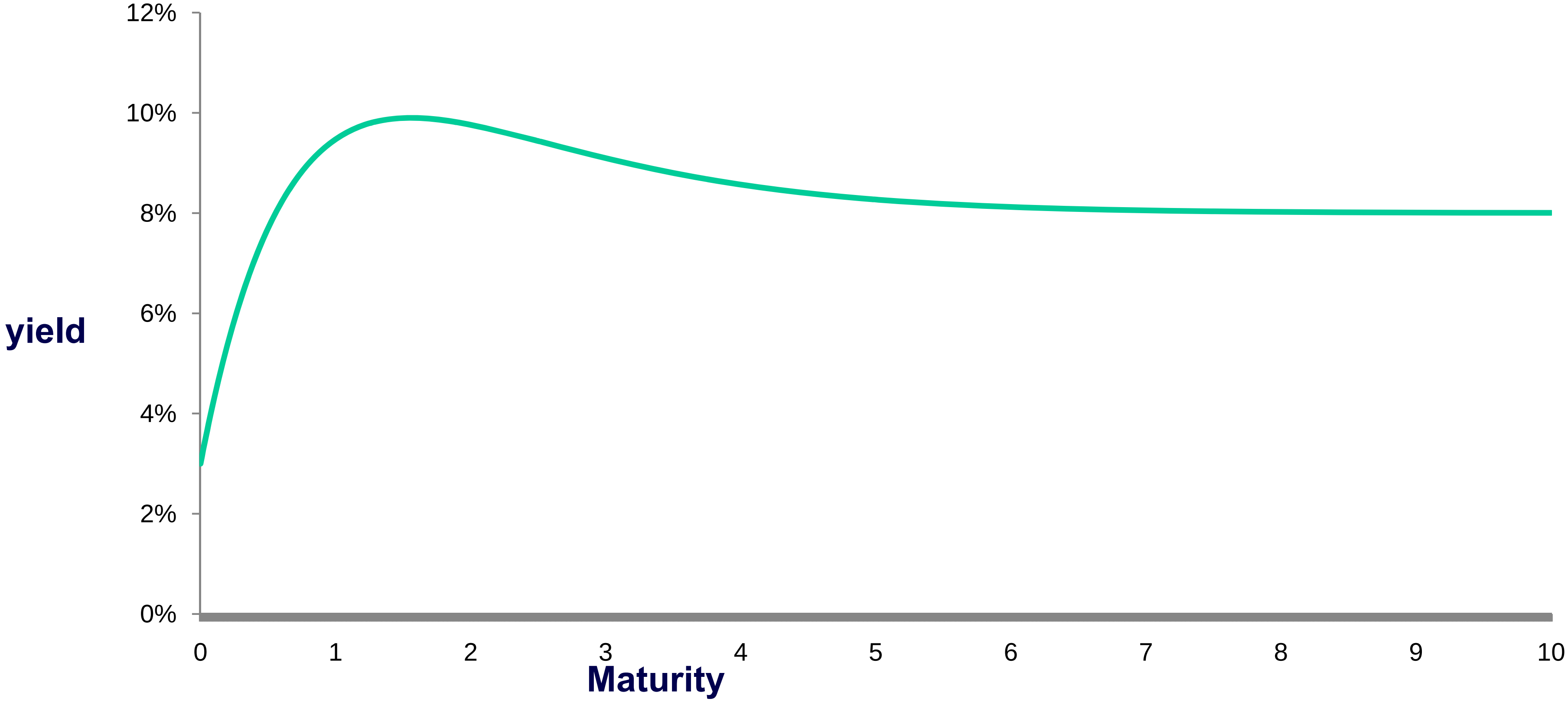
with appropriate 'weights' for these components, the model can generate a variety of forward rate curves with monotonic and humped shapes.

Interpretation of model parameters



$\beta_0 = 0.09,\ \beta_1 = -0.06,\ \beta_2 = 0.02$

Interpretation of model parameters



$$\beta_0 = 0.08, \beta_1 = -0.05, \beta_2 = 0.09$$

Section 2: Methodology

Regression setup

OLS formulation

Regression setup

$$R(m) = \beta_0 + (\beta_1 + \beta_2)^* (1 - \exp(-m/\tau)) / (m/\tau) - \beta_2^* \exp(- m/\tau)$$

- For a particular value of τ, this becomes a linear model of the form-

$$R(m) = a + bx_1 + cx_2$$

$$\text{Where } x_1 = (1 - \exp(-m/\tau))/ (m/\tau) \quad \text{and } x_2 = \exp(-m/\tau)$$

- For a given value of τ, the linear model can be fitted using Ordinary Least Squares (OLS) regression to obtain the best-fitting values of a, b and c.
- Then a grid-search mechanism for τ, i.e., repeating this procedure across a pre-determined range of values of τ gives us the best over-all fit.

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OLS formulation

- Due to collinearity, it is inappropiate to perform regression using standard statistical software

- instead, the regression should be performed algebraically from the first principles.

- Suppose the market data includes n yields R_1, R_2,… R_n corresponding to maturities m_1, m_2, … m_n
- Having fixed τ, the objective is to determine parameters a, b and c that minimize the total sum of squares of errors.

OLS formulation

To minimise $S = \sum (R_i - a - bx_{1i} - cx_{2i})^2$ w.r.t. a,b and c

$\partial S/\partial a = 0 \quad \Rightarrow \quad na + b\sum x_{1i} + c\sum x_{2i} = \sum R_i$

$\partial S/\partial b = 0 \quad \Rightarrow \quad a\sum x_{1i} + b\sum x_{1i}^2 + c\sum x_{2i}\, x_{1i} = \sum R_i\, x_{1i}$

$\partial S/\partial c = 0 \quad \Rightarrow \quad a\sum x_{2i} + b\sum x_{2i}\, x_{1i} + c\sum x_{2i}^{\,2} = \sum R_i\, x_{2i}$

in matrix formulation, $A\beta = R$

Where β is the vector (a, b, c), matrix A contains the corrs. coefficients, and R is the vector of the right-hand side values in the three eqns.

The solution is given by

$$\beta = A^{-1} R$$

Section 3: Issues

3.1 Fixing tau: motivation and implication

3.2 Collinearity of regressors

3.3 Specifying the range of values of tau

3.4 Data issues and the 30 year yield

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3.1 Why fix tau?

- **Primarily because it transforms**

  $R(m) = \beta_0 + (\beta_1 + \beta_2)^* (1 - \exp(-m/\tau)) / (m/\tau) - \beta_2^* \exp(- m/\tau)$

  into an easily tractable linear model

  $$R(m) = a + bx_1 + cx_2$$

- **Grid search mechanism to determine tau in a specified range of values**

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3.1 The implications of fixing tau

- In our model τ is the scaling parameter
- τ does not affect the curve-fitting very much, but it largely determines the shape of the yield curve, particularly at higher maturities
- A particular value of τ can give a marginally better fit than other values, but an extremely 'ugly' curve otherwise
- Fixing τ within a range beforehand is a control on such erratic results

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3.1 An interpolation analogy

- **Yield curve construction from market data can be split into two parts – interpolation within the range of maturities in the data, and extrapolation for higher maturities**
- **In our model (approximately speaking) parameters β_0, β_1 and β_2 represent the interpolation part, while τ represents the extrapolation part or the 'shape' aspect of the construction**
- **Fixing τ thereby implies putting a handle on the curve shape and then choosing the best fit ('interpolation') that gives the desired shape**

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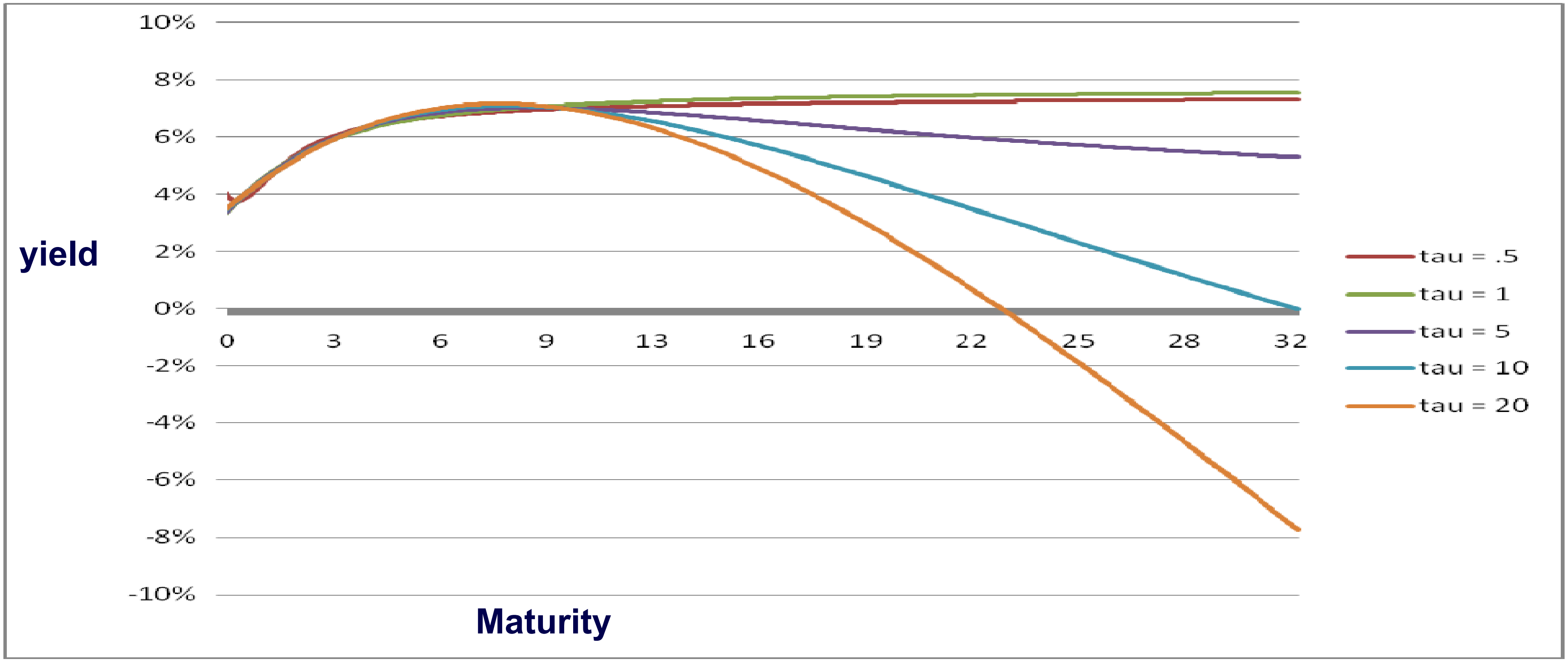
3.1 An illustration



ZCYC for 30th June 2009

The curves are close till maturity of 10, i.e. the in-sample part (we calibrated the curve with market data on bonds of up to 10 yrs maturity). However, they move in totally different directions for higher maturities. Thus, the value of tau largely determines the shape of the curve but is not very crucial to the data-fitting part.

3.2 Why collinearity arises

$$R(m) = a + bx_1 + cx_2$$

where $x_1 = (1 - \exp(-m/\tau))/(m/\tau)$ and $x_2 = \exp(-m/\tau)$

Let $y = m/\tau$ and consider the function $f(y) = -\exp(-y)$

$f'(y) = \exp(-y)$

$(f(y) - f(0))/y = (1 - \exp(-y))/y$

(The first quantity is x_1 and the second quantity is x_2)

- As y approaches zero, both these quantities approach $f'(0)$
- When m is small, or when τ is large, then y is close to zero, and hence we have high correlation between x_1 and x_2.

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3.2 How this might affect our regression

Let us assume that the correct model is

$$R(m) = 1 + 2x_1 + 3x_2 \text{ .. (1)}$$

As $x_1 - x_2$ is close to zero,

$$R(m) = 1 + 3x_1 + 2x_2 \text{ .. (2)}$$

may also give a very good fit and when the regression is performed directly, this might be the model that is obtained as the best fit.

- However, though (2) and (1) are very similar at small maturities, they are very different at high maturities (as the correlation between x_1 and x_2 vanishes at high values of m).
- Hence the yield curve obtained from (2) is a grossly incorrect one.

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3.2 How to deal with this

- It is preferable to determine the coefficients algebraically from the first principles of OLS regression, instead of using some statistical software
- Range of values of τ are specified such that extreme collinearity does not arise, making the regression more robust.

3.3 Specifying the range of values for tau

The fitting aspect-

- **A small value of τ implies higher 'flexibility' at small values of m, and vice versa**
- **Keeping in mind our range of maturities, we should look for values of tau such the model gives a good fit 'on average'**
- **It is assumed that the data includes bonds of the following maturities – 3 months, 6 months, 1 year, 2 years, 3 years, 4 years, 5 years and 10 years**

3.3 Specifying the range of values for tau

- ***The collinearity aspect-***

value of tau	collinearity of regressors
0.25	0.9377
0.5	0.9586
1	0.9707
2	0.9803
5	0.9942
10	0.9985
20	0.9996

Thus, collinearity reaches extreme levels for tau = 5 onwards

- **Keeping in mind the mix of maturities in our data, the desired stability of shape of the yield curve, and the collinearity of regressors, we recommend using [0.3, 2] as the range of values of tau, in increments of 0.1.**

3.4 Data used for curve fitting

- We have used market zero coupon yields published by Bloomberg
- Bloomberg publishes daily zero-coupon yields data for the following maturities –

  3 months, 6 months, 1 yr, 2 yr, 3yr, 4yr, 5yr, 10yr, 30yr
- These annualized yields have to be converted to their continuously compounded equivalents
- It is recommended not to use 30 yr yields in the data

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3.4 Issues with 30 yr yields

Five month-end curves were constructed including 30 year yields

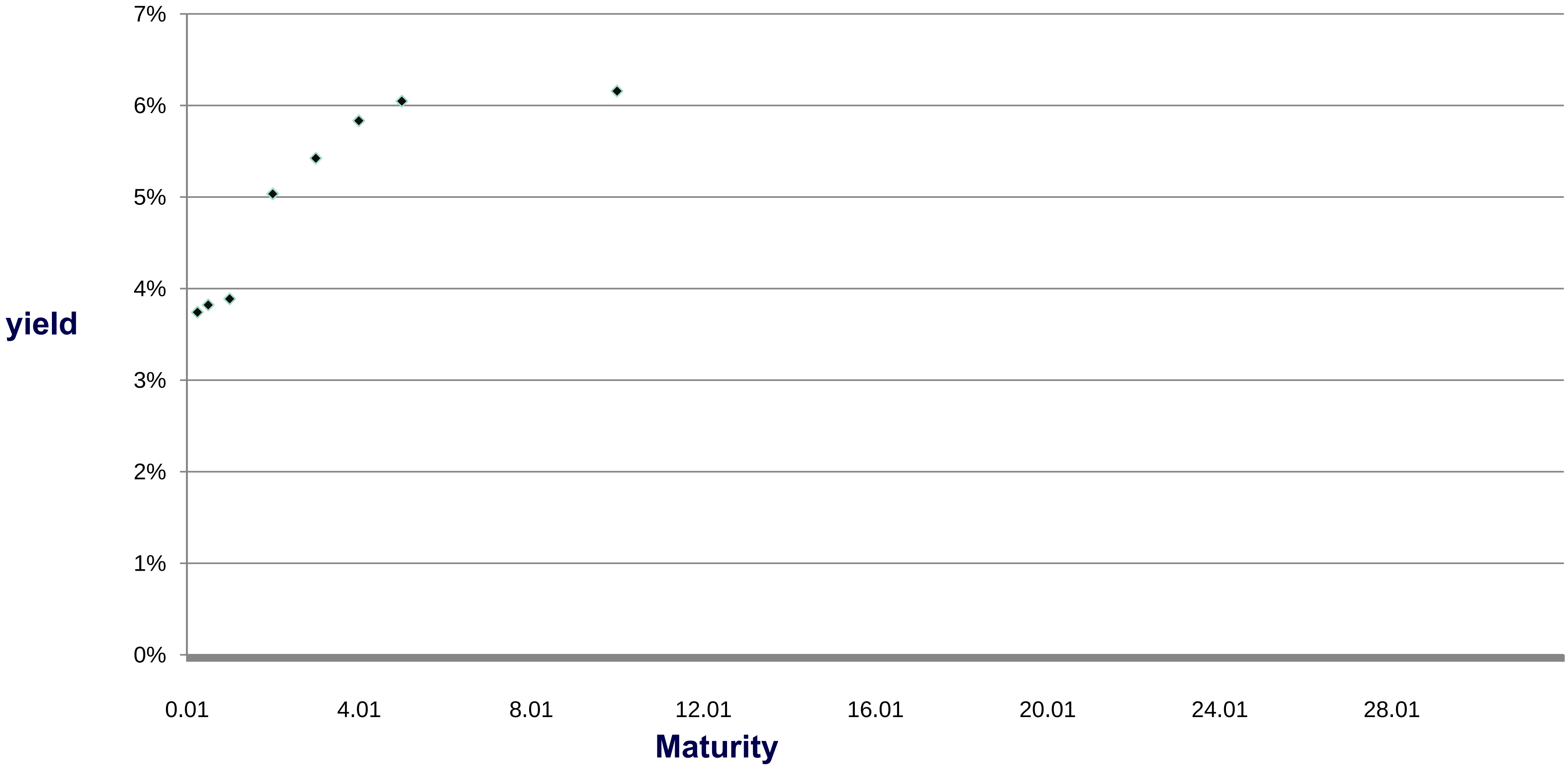
month	Bloomberg	Model 1	Model 1 – Bloomberg	Model 1 sd	Model 2	Model 2 - Bloomberg
Feb-09	8.06%	6.23%	-1.83%	0.13%	7.58%	-0.48%
Mar-09	7.74%	7.30%	-0.44%	0.10%	7.57%	-0.17%
Apr-09	7.45%	6.49%	-0.96%	0.10%	7.04%	-0.41%
May-09	7.62%	7.07%	-0.55%	0.10%	7.40%	-0.22%
Jun-09	7.86%	7.29%	-0.57%	0.06%	7.68%	-0.18%

Model 1 refers to the fitted curve with data excluding 30 yr yields

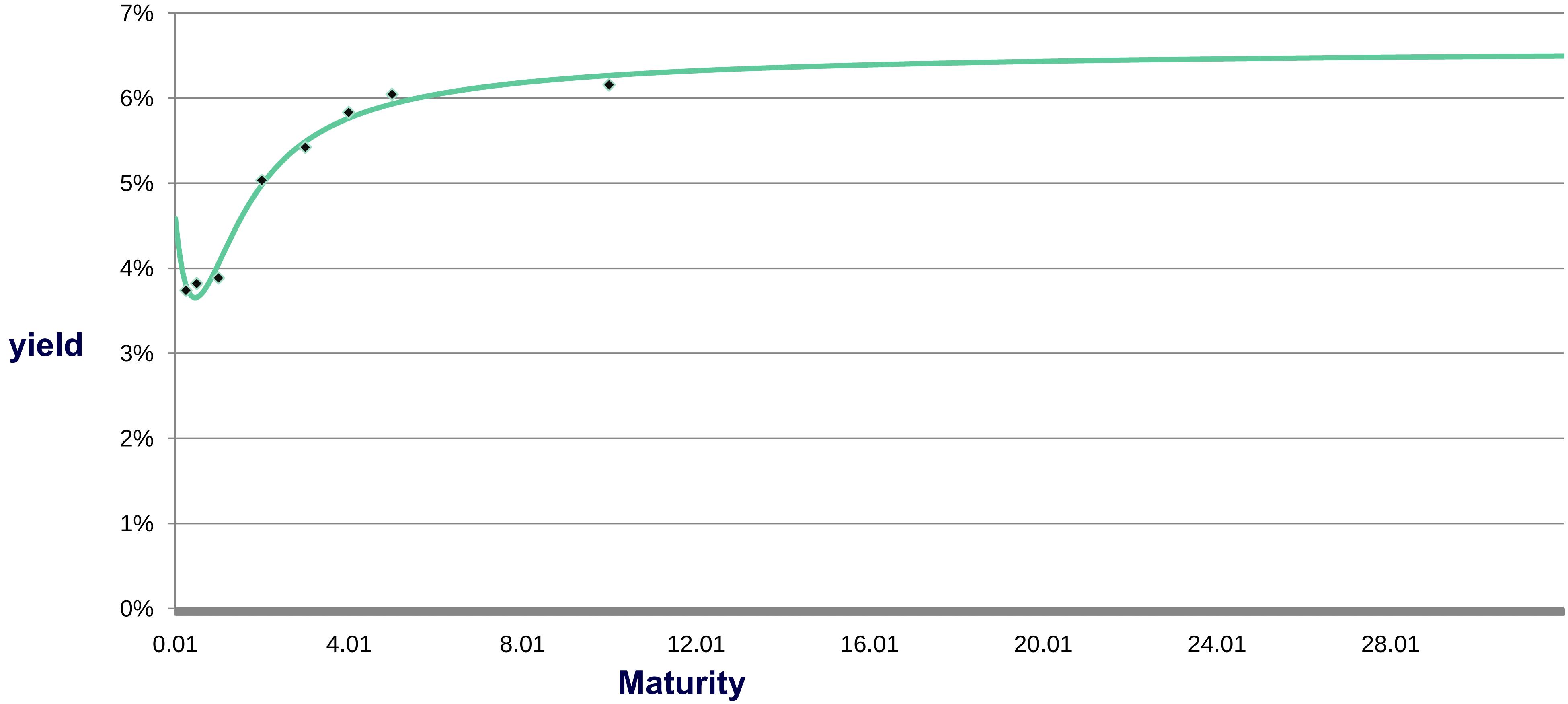
Model 2 refers to the fitted curve with data including 30 yr yields

- **Model 1 consistently underestimates 30 yr yields, to an extent that is severe when compared to sd (which represents average error)**
- **This implies 30 yr yields are much higher than what is expected from prevalent yields on other maturities**
- **Even when 30 yr yields are included in the data (i.e. in model 2), it is consistently underestimated**

3.4 A visual illustration

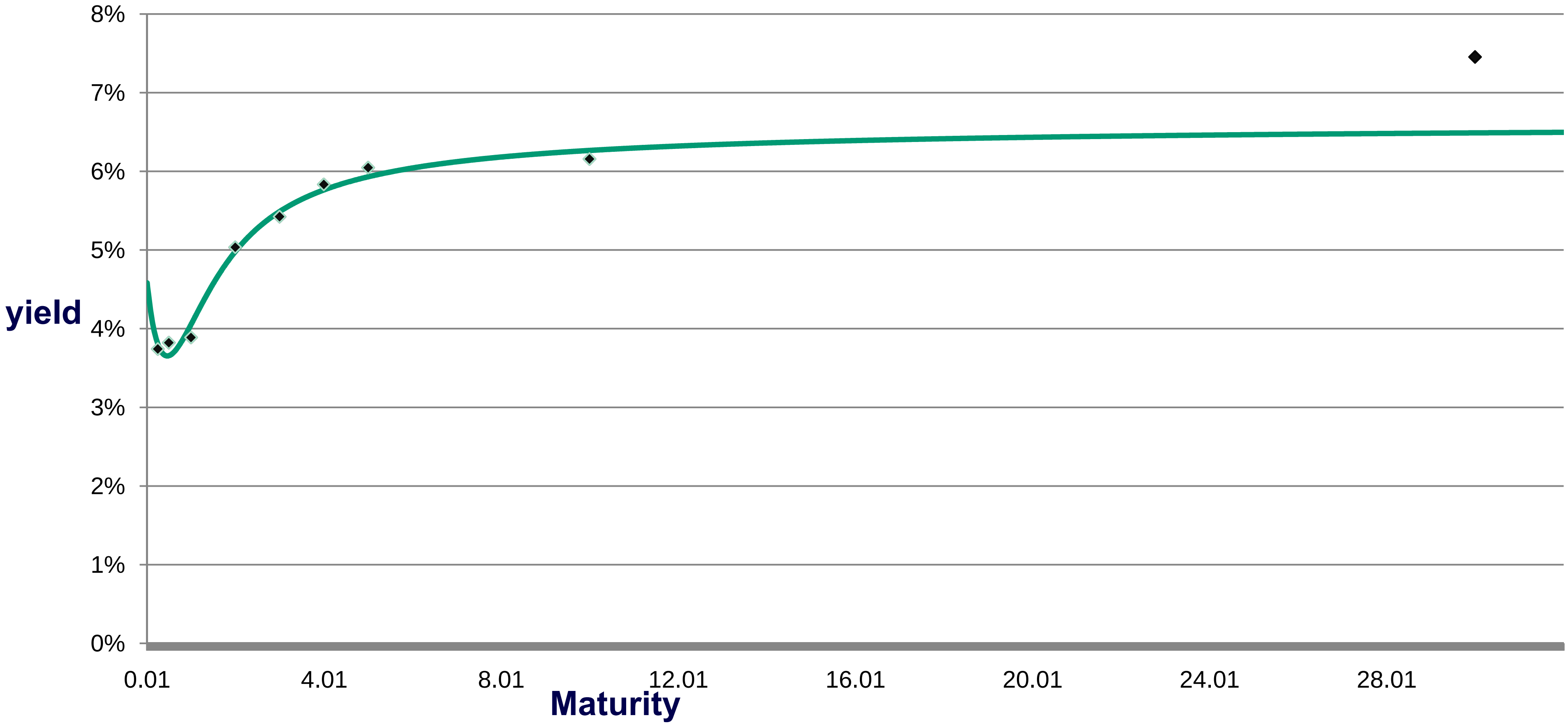


3.4 A visual illustration



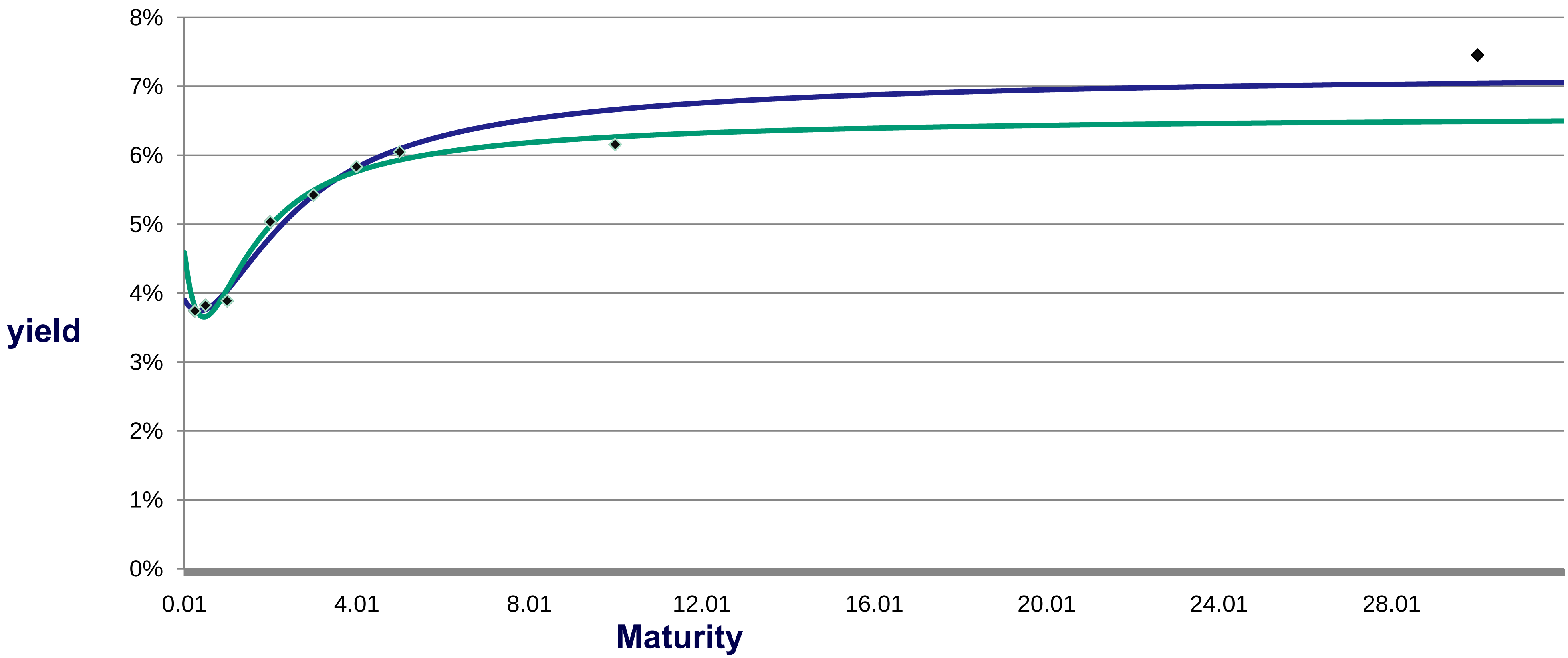
Bloomberg yields (till 10 yr) as on April 2009 and the fitted curve

3.4 A visual illustration



- **The 30 year yield is way above the yield curve**

3.4 A visual illustration



- **The 30 yr yield 'pulls up' the yield curve, but still there is a significant negative error**
- **We interpret this as liquidity premium in 30 yr yields**
- **This is further emphasized from volume-wise G-Sec trading data at CCIL**

VOLUME-WISE OUTRIGHT TRADE DETAILS IN GOVERNMENT SECURITIES AND
TREASURY BILLS RECEIVED BY CCIL FOR SETTLEMENT
FROM 10-Feb-10 TO 10-Feb-10

ISIN No.	Security Description	No Of Trades	Maturity Date	Volume (Rs. Crs)	Price(Rs.) High	Price(Rs.) Low	Price(Rs.) Weighted Average	Yield(%) High	Yield(%) Low	Yield(%) Weighted Average
Central Government Securities										
IN0020020171	6.35% G.S 2020	669	02-Jan-20	4840.50	90.87	90.07	90.37	7.8080	7.6838	7.7610
IN0020090059	7.02% GOVT.STOCK 2016	311	17-Aug-16	2583.00	97.50	97.06	97.28	7.6020	7.5128	7.5566
IN0020090026	6.49% GOVT.STOCK 2015	41	08-Jun-15	510.00	95.49	95.16	95.31	7.6116	7.5306	7.5737
IN0020090042	6.90% GOVT.STOCK 2019	58	13-Jul-19	415.08	93.88	93.26	93.69	7.9292	7.8291	7.8767
IN0020090067	7.32% GOVT.STOCK 2014	16	20-Oct-14	315.00	100.50	100.32	100.45	7.2344	7.1887	7.2008
IN0020092071	GOI FLOATING RATE BOND 2020	13	21-Dec-20	110.00	92.90	92.80	92.87	4.6410	4.6286	4.6326
IN0020020064	7.55% G.S. 2010	5	14-May-10	85.50	100.90	100.87	100.90	4.1740	3.9703	3.9977
IN0020089051	6.65% FERT CO GOISPL BOND 20	2	29-Jan-23	78.44	87.25	87.25	87.25	8.2709	8.2709	8.2709
IN0020070028	8.08% GOVT.STOCK 2022	5	02-Aug-22	75.00	100.10	100.05	100.07	8.0731	8.0667	8.0701
IN0020010073	9.40% GS 2012	2	11-Sep-12	45.00	107.05	107.05	107.05	6.3918	6.3918	6.3918
IN0020020106	7.95% G.S 2032	8	28-Aug-32	29.00	97.50	95.96	96.17	8.3510	8.1942	8.3285
IN0020010107	8.07% GS 2017	1	15-Jan-17	25.00	101.75	101.75	101.75	7.7371	7.7371	7.7371
IN0020070051	8.13% GOVT.STOCK 2022	3	21-Sep-22	25.00	100.22	100.22	100.22	8.1000	8.1000	8.1000
IN0020020130	7.38% G.S 2015	3	03-Sep-15	15.00	100.00	100.00	100.00	7.3784	7.3784	7.3784
IN0020060078	8.24% GOVT.STOCK 2027	4	15-Feb-27	12.00	99.51	99.30	99.32	8.3174	8.2941	8.3161
IN0020020056	7.40% G.S. 2012	2	03-May-12	10.00	102.65	102.61	102.63	6.1222	6.1030	6.1126
IN0020060045	8.33% GOVT.STOCK 2036	2	07-Jun-36	10.00	99.85	99.80	99.83	8.3470	8.3423	8.3447
IN0020080019	8.24% GOVT.STOCK 2018	1	22-Apr-18	5.00	102.00	102.00	102.00	7.9007	7.9007	7.9007
IN0020030097	5.64% GOVT. STOCK 2019	2	02-Jan-19	4.00	86.05	86.00	86.03	7.8560	7.8473	7.8517
IN0020060052	8.01% OIL MKT COS GOI SB 2023	4	15-Dec-23	1.06	98.45	98.24	98.28	8.2232	8.1972	8.2188
IN0020050012	7.40% GOVT.STOCK 2035	1	09-Sep-35	0.50	90.40	90.40	90.40	8.3104	8.3104	8.3104
IN0020089010	8.20% OIL MKN GOISPL BOND 202	1	10-Nov-23	0.30	99.70	99.70	99.70	8.2343	8.2343	8.2343
IN0020099019	8.20% OIL MKTG COS GOI SB 202	1	15-Sep-24	0.25	99.80	99.80	99.80	8.2223	8.2223	8.2223
	Sub Total :	1155		9,194.63						

- **There are negligible volumes traded at maturities close to 30 yrs**
- **As our purpose is to construct a 'risk-free' curve, we don't want to include liquidity premium**
- **Hence 30 yr yields are not used for ZCYC calibration**

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Section 4: Results

4.1 Yield errors

4.2 Price errors

4.3 Stability of shape

4.1 Yield errors

month	Yield error for fitted curve	Yield error for NSE ZCYC
Dec-08	0.05%	1.56%
Jan-09	0.15%	0.67%
Feb-09	0.10%	0.39%
Mar-09	0.09%	0.23%
Apr-09	0.10%	0.54%
May-09	0.10%	0.73%
Jun-09	0.06%	0.88%
Jul-09	0.08%	0.58%
Aug-09	0.09%	0.14%
Sep-09	0.16%	0.30%
Oct-09	0.17%	0.30%
Nov-09	0.06%	0.74%

- **Compared model yields and Bloomberg yields, and considered the average yield error across the range of maturities. The same was obtained for the NSE ZCYC.**
- **The yield curve obtained by our methodology was found to give lower average errors consistently.**

4.2 Price errors

Month	Fitted curve for best tau	Nse ZCYC
Dec-08	0.15	1.79
Jan-09	0.35	1.54
Feb-09	0.34	0.92
Mar-09	0.22	0.40
Apr-09	0.22	0.86
May-09	0.25	0.94
Jun-09	0.10	0.81
Jul-09	0.20	0.72
Aug-09	0.20	0.37
Sep-09	0.39	0.46
Oct-09	0.43	0.64
Nov-09	0.10	0.71

- **From model yields we calculated prices for zero coupon bonds of face value 100/- and these were compared to prices obtained from Bloomberg market yields. The same was done for yields obtained from the NSE ZCYC.**
- **The price errors from our fitted yield curve were significantly lower than those from the benchmark NSE yield curve.**

4.3 Stability of shape

- **Moreover, the curves obtained were of stable shape.**
- **In fact, in the Nelson Siegel model, τ is the scaling parameter and hence largely determines the shape of the curve, particularly for higher maturities. Hence this methodology implicitly ensures that shape of the curve doesn't change widely, as we are restricting τ to the range [0.3, 2].**
- **The following charts show shapes of the last four quarter-end yield curves.**

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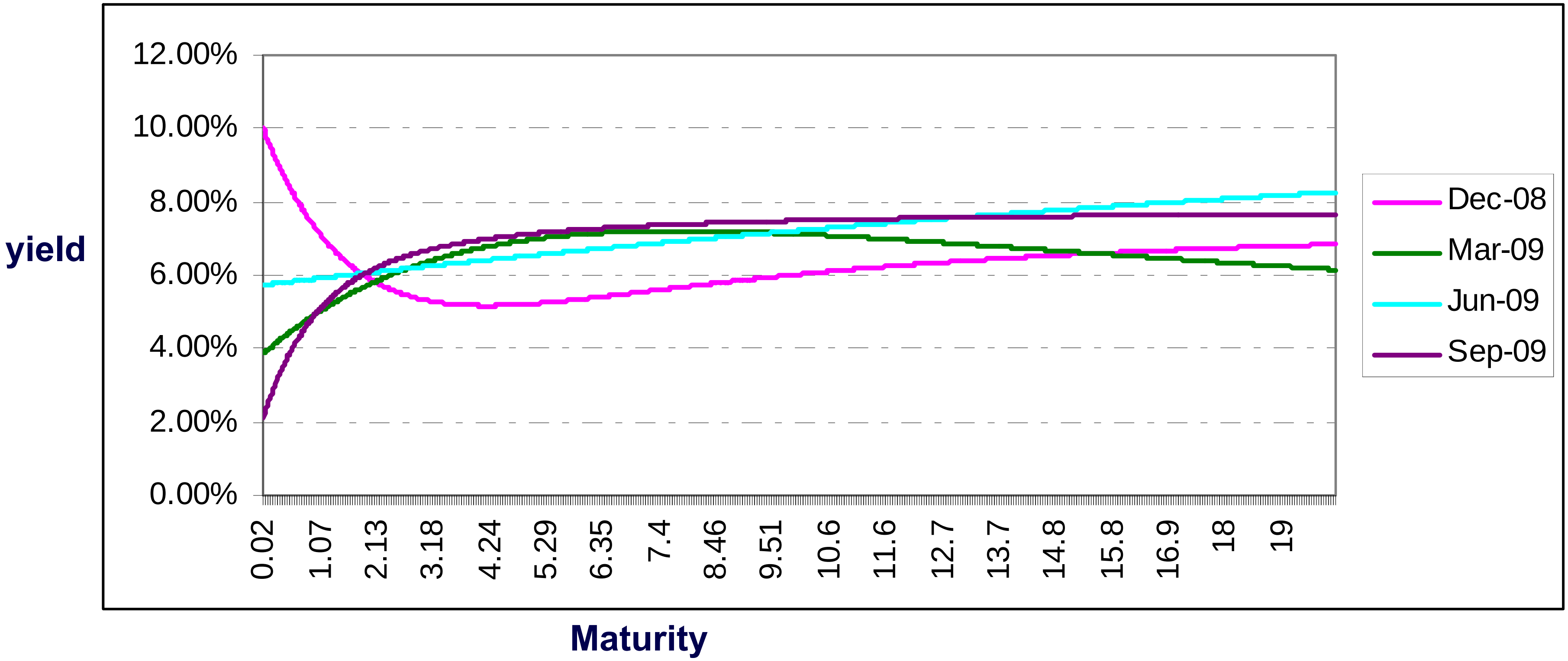
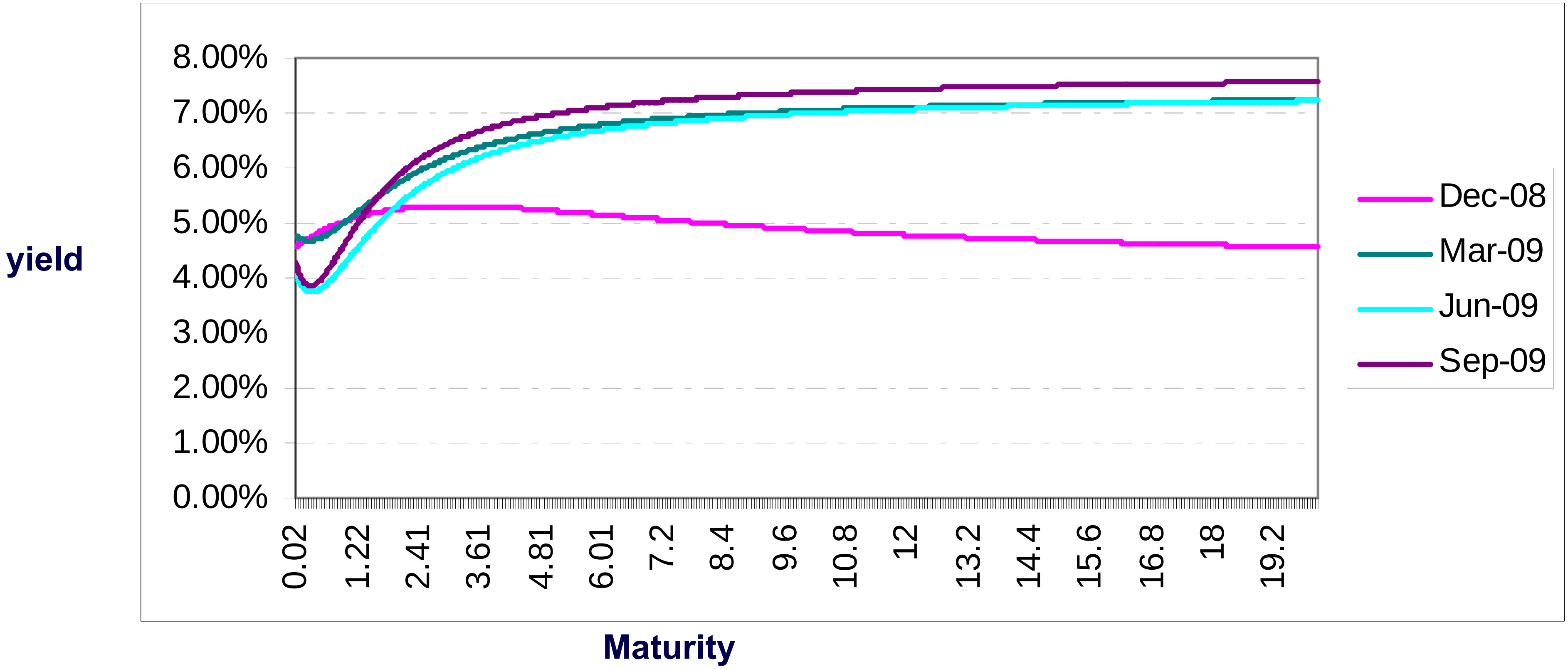
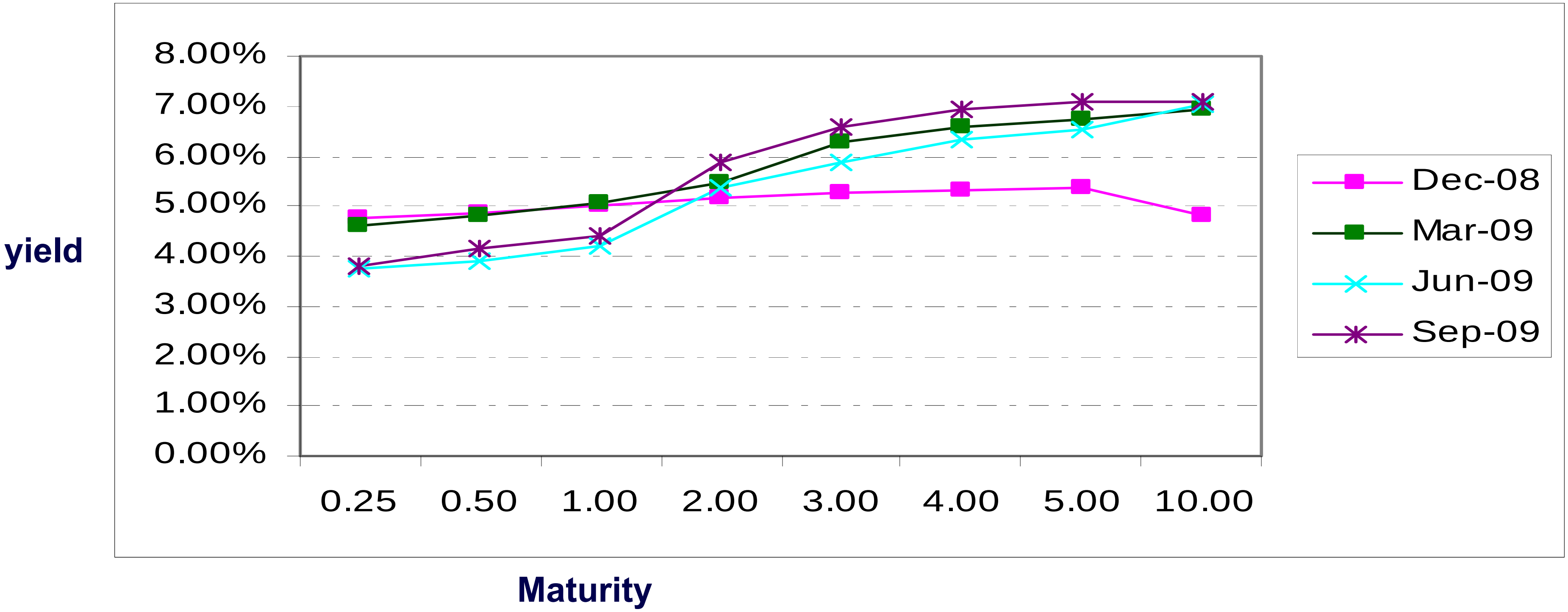
4.3 NSE yield curves



4.3 Fitted yield curves



4.3 Bloomberg yields



Comparing the three charts shows that the fitted yield curve has better stability of shape, and can capture actual market yield dynamics better than the NSE ZCYC.

Conclusion

- The model consistently outperforms the benchmark NSE yield curve
  - Better fit to market yields and prices
  - Improved stability of shape
- This methodology can be easily applied for an in-house calibration of the yield curve from market data
- We have identified liquidity premium in 30 year zero coupon yields

Sources

- **Papers:**
- **Parsimoniuos modeling of yield curves – by Charles R. Nelson and Andrew F. Siegel, The Journal of Business, Vol. 60, No. 4. (Oct., 1987)**
- **Forecasting the term structure model of government bond yields – by Francis X. Diebold and Canlin Li, Journal of Econometrics (2006)**
- **Data:**
- **Zero coupon yields from Bloomberg**
- **Historical ZCYC from archives at nseindia.com**
- **G-Sec trading data from** *www.**ccilindia**.com*

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