Institute of Actuaries of India

Subject CT8 – Financial Economics

September 2018 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) The 'excessive volatility' arises when the security prices are more volatile than the underlying fundamental variables that should be driving them and when the change in prices couldn't be justified by the news arriving alone. This was claimed to be evidence of market over-reaction which was not compatible with the efficient market hypothesis. Under EMH, all the information is expected to reflect fully under the security market prices and any deviations are assumed to be the result of 'under' or 'over' reaction to events. [1.5]

ii) The claim of 'excessive volatility' was first put into testable proposition by Shiller in 1981. He used a discounted cash flow model based on the actual dividends that were paid and some terminal value for the share to calculate a perfect foresight price for the equity. This would represent the "correct" equity price if market participants had been able to predict future dividends correctly.

The difference between the perfect foresight price and the actual price arises from the forecast errors of future dividends. If market participants are rational, there should be no systematic forecast errors. Also if markets are efficient, then broad movements in the perfect foresight price should be correlated with moves in the actual price as both are reacting to the same news and hence the same changes in the anticipated future cash flows.

However, he found strong evidences that the actual movements and hence the observed level of volatility contradicted the EMH, i.e. the markets are 'excessively volatile'. [2.5]

iii) Subsequently, the approach used by Shiller was criticized due to various reasons such as:

- the difficulty of choosing an appropriate terminal value for the share price
- choosing the discount rate with which to discount future cash flows – in particular, should it be constant
- possible biases in the estimates of the variances because of autocorrelation in the time series data used
- possible non-stationarity of the time series data used, *ie* it may have stochastic trends which invalidate the measurements obtained for the variance of the stock price
- the distributional assumptions underlying the statistical tests used might not be satisfied

Later, many authors tried to overcome the above limitations, but no work has been concluded without dividends and distributional assumptions which were questioned earlier. Hence, the vast literature of volatility remains inconclusive. [3]

[7 Marks]

Solution 2:

i) In the application of utility theory to finance and investment choices, it is assumed that a numerical value called the utility can be assigned to each possible value of the investor's wealth by what is known as a utility function.

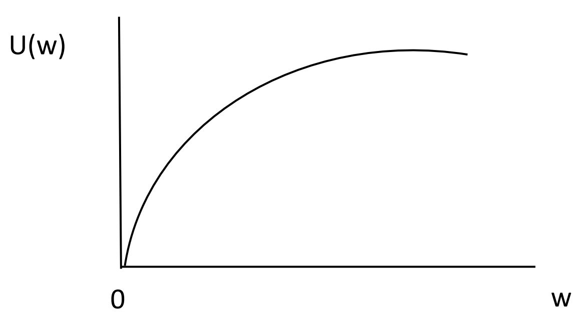
The utility function, u(w) of any individual is usually influenced by the risk-return preferences and how they behave in normal course of action. The shape in particular is influenced by: [1]

ii)

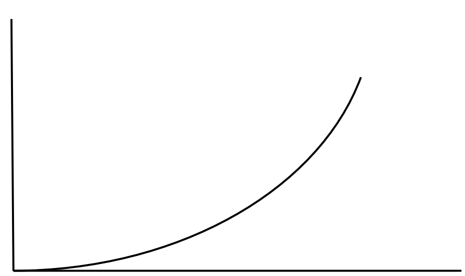
a) **Non Satiation**: It is commonly assumed that people prefer more wealth to less and hence are non-satiated. This also suggests that the marginal utility of wealth is strictly positive and is expressed as:
$U'(w) > 0$
So the curve of utility function should be upward sloping.

b) **Risk aversion**: The attitude towards risk of every individual has influence on the utility and is reflected in the choices made by him. The investors are usually classified into 3 main categories:

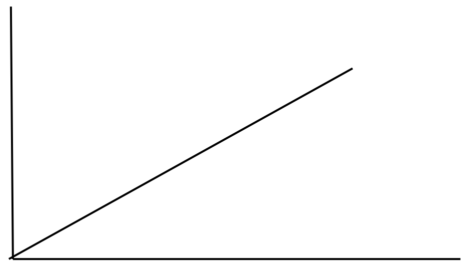
- **Risk Averse**: An investor who values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. His marginal utility is positive but keeps on decreasing with every money earned, i.e. $U''(w) < 0$.
-



- **Risk-seeking**: An investor who values an incremental increase in wealth more highly than an incremental decrease and will seek a fair gamble. For him, $U''(w) > 0$



- **Risk-neutral**: An investor who is indifferent to accepting fair gambles and status quo, $U''(w) = 0$. His marginal utility remains constant with every additional wealth.



[4]

iii) Some commonly used utility functions are:

1. **Quadratic**: This is in the form of $U(w) = a + bw + cw^2$ or $U(w) = w + dw^2$. The calculation of U'(w) and U''(w) suggests that the function exhibits both increasing absolute and relative risk aversion. In other words, it is consistent with an investor who keeps an increasing proportion of wealth in risky assets as she gets richer in addition to an increasing absolute amount invested in risky assets.

2. **Log**: This is in the form of U(w) = ln (w), which exhibits declining absolute risk aversion and constant relative risk aversion. This is consistent with an investor who keeps a constant proportion of wealth invested in risky assets as they get richer, in addition to an increasing absolute amount of wealth invested in risky assets.

3. **Power**: This is in the form of $U(w) = w^{\Upsilon} - 1/\Upsilon$ and like log function, this also suggests declining absolute risk aversion and constant relative risk aversion.

[3]

[8 Marks]

Solution 3:

i) The risk neutral probability for Gold, p_{gold} =
$1 = (1 - p_{gold}) * 1.06/1.04 + p_{gold} * 0$
$p_{gold} = 1.8868\%$

similarly, $p_{silver} = 3.7037\%$ [1]

ii) Now, **for Investor X**, 95% VaR (Value at Risk) is zero.
95% Tail VaR = $1.06\ p_{gold} / p_{gold} = 1.06$

For Investor Y, again, 95% VaR is zero.
95% Tail VaR = $1.08\ p_{silver} / p_{silver} = 1.08$

For Investor Z, either one can default or both can default and both cannot default. So the distribution of returns would be –

1.07 with probability $(1 - p_{gold})(1 - p_{silver}) = 0.94479$
0.54 with probability $p_{gold}(1 - p_{silver}) = 0.01817$
0.53 with probability $(1 - p_{gold})\, p_{silver} = 0.03634$
0 with probability $p_{gold}\, p_{silver} = 0.00070$

So 95% VaR is 1.07 – 0.54 = 0.53

The 95% Tail VaR is

$$\frac{1.07 * p_{silver} * p_{silver} + 0.54 (1 - p_{silver})\, p_{silver}}{p_{silver}} = 0.55$$

[4]

iii) **Comments on results**:
Usually, investment in diversified portfolio leads to a lower dispersion of returns and hence lower risk. In the example above, Investor Z invested in a diversified portfolio compared to X and Y but his VaR is higher than for either X or Y. So the increase in VaR could correspond to a decrease in risk under such circumstances.
Further, zero VaR does not necessarily mean zero risk.
As expected, the tail VaR for Investor Z is lower than the Investor X and Y.

Comments on appropriateness of VaR and Tail VaR as measures of investment risk:
VaR represents the maximum potential loss on a portfolio over a given future time period with a given degree of confidence. It is often calculated assuming that investment returns follow a normal distribution, which may not be an appropriate assumption.
The usefulness of VaR in case of non-normal distributions depend on modelling skewed or fat-tailed distribution of returns. The further one gets into the "tails" of the distributions, the more lacking the data and hence, the more arbitrary the choice of the underlying probability distribution becomes.

TailVaR measures the expected loss in excess of the VaR, hence, relative to VaR, it provides much more information on how bad returns can be when benchmark level is exceeded. It has the same modeling issues as VaR in terms of sparse data, but captures more information on tail of the non-normal distribution.

[4]

[9 Marks]

Solution 4:

i) The given relationship can be written as:

$S_t = S_0\, e^{\mu t + \sigma B t}$

Since S_t is a function of standard Brownian motion, B_t, applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$dB_t = 0 X dt + 1 X dB_t$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B t}$, then
$dG/dt = \mu S_0 e^{\mu t + \sigma Bt} = \mu S_t$
$dG/dB_t = \sigma S_0 e^{\mu t + \sigma Bt} = \sigma S_t$
$d^2G/dB^2_t = \sigma^2 S_0 e^{\mu t + \sigma Bt} = \sigma^2 S_t$

Hence, using Ito's Lemma from Page 46 in the Tables we have:
$dG = [0 X \sigma S_t + ½ X 1^2 X \sigma^2 S_t + \mu S_t] dt + 1 X \sigma S_t dB_t$

i.e. $dS_t = (\mu + ½ \sigma^2) S_t dt + \sigma S_t dB_t$

Thus,
$dS_t/S_t = \sigma dB_t + (\mu + ½ \sigma^2) dt$

So, $c_1 = \sigma$ and $c_2 = \mu + ½ \sigma^2$ [4]

ii) The expected value of S_t is:
$E[S_t] = E [S_0 e^{\mu t + \sigma Bt}] = S_0 e^{\mu t} E[e^{\sigma Bt}]$

Since $B_t \sim N (0,1)$, its MGF is $E[e^{\Theta Bt}] = e^{½ \Theta 2t}$
So, $E[S_t] = S_0 e^{\mu t} X e^{½ \sigma 2t} = S_0 e^{\mu t + ½ \sigma 2t}$

The variance of S_t is:
$Var[S_t] = E[S^2_t] – (E[S_t])^2$
$= E[S^2_0 e^{2\mu t +}{}_{2\sigma Bt}] – (S_0 e^{\mu t + ½ \sigma 2t})^2$
$= S^2_0 e^{2\mu t} E[e^{2\sigma Bt}] – S^2_0 e^{2\mu t + \sigma 2t}$
$= S^2_0 e^{2\mu t + 2\sigma 2t} - S^2_0 e^{2\mu t + \sigma 2t}$
$= S^2_0 e^{2\mu t} (e^{2\sigma 2t} - e^{\sigma 2t})$ [4]

iii) $Cov[S_{t1}, S_{t2}] = E[S_{t1}, S_{t2}] - E[S_{t1}] E[S_{t2}]$
From above,
$E[S_{t1}] = S_0 e^{\mu t1 + ½ \sigma 2t1}$ and $E[S_{t2}] = S_0 e^{\mu t2 + ½ \sigma 2t2}$

The expected value of the product is:
$E[S_{t1}, S_{t2}] = E[S_0 \exp(\mu t_1 + \sigma B_{t1}) S_0 \exp (\mu t_2 + \sigma B_{t2})]$

$= S^2_0\ e^{\mu(t1 + t2)}\ E[\exp(\sigma B_{t1} + \sigma B_{t2})]$

To evaluate this we need to split B_{t2} into two independent components:
$B_{t2} = B_{t1} + (B_{t2} - B_{t1})$ where $B_{t2} - B_{t1} \sim N(0, t_2 – t_1)$

Hence,
$E[S_{t1}, S_{t2}]$
$= S^2_0\ e^{\mu(t1 + t2)}\ E[\exp(\sigma B_{t1} + \sigma \{ B_{t1} + (B_{t2} - B_{t1})\})]$
$= S^2_0\ e^{\mu(t1 + t2)}\ E[\exp(2\sigma B_{t1} + \sigma \{ B_{t2} - B_{t1}\})]$
$= S^2_0\ e^{\mu(t1 + t2)}\ E[\exp(2\sigma B_{t1})]\ E[\exp \{ B_{t2} - B_{t1}\})]$
$= S^2_0\ e^{\mu(t1 + t2)}\ \exp(2\sigma^2 t_1)\ \exp [\ ½\ \sigma^2\ (t_2 – t_1)]$
$= S^2_0\ e^{\mu(t1 + t2)}\ \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2)$

Putting all the equations together:

$\mathrm{Cov}[S_{t1}, S_{t2}] = S^2_0\ e^{\mu(t1 + t2)}\ \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2) - S_0\ e^{\mu t1 + ½\ \sigma 2t1}.\ S_0\ e^{\mu t2 + ½\ \sigma 2t2}$
$= S^2_0\ e^{\mu(t1 + t2)}\ (\exp(\frac{3}{2}\sigma^2 t_1\ ) – \exp(\frac{1}{2}\sigma^2 t_1))\ \exp(\ ½\ \sigma^2 t_2)$ [6]

[14 Marks]

Solution 5:

i) Factors are:

- the underlying share price, S_t
- the strike price, K
- the time to expiry, T –t
- the volatility of the underlying share, σ
- the risk-free interest rate, r
- Dividend rate, if any, q [2]

ii) Impact on value of American options:

Factors (Increase)	Call	Put
Share Price	Increase	Decrease
Strike Price	Decrease	Increase
Time to expiry	Increase	Increase
Volatility	Increase	Increase
Risk Free Interest rate	Increase	Decrease
Dividend rate	Decrease	Increase

[4]

iii) Arbitrage strategy

As the outcome will be certain that only one country will win, the value of arbitrage opportunity arising due to betting is as under:
V_1 = 1/1.5 +1/2.4 = 1.08
V_2 = 1/ 1.7 + 1/3.0 = 0.92
As $V_1 > 1$, hence the arbitrage opportunity does not exist. The arbitrage opportunity arises with V_2 strategy as its value is less than 1.

Arbitrage profit = 1 – .92= 8%

The amount to bet on Francisco for 1.7 odd:
(1000/0.92)*(1/1.7) = Rs 640
The amount to bet on Croatiano for 3.0 odd:
Rs 1000 - Rs 640 = Rs 360

Outcome	Result	Profit
Francisco wins	Rs 640 * 1.7 = 1088	Rs 88
Croatiano wins	Rs 360 * 3.0 = 1088	Rs 88

[4]
[10 Marks]

Solution 6:

i) Under the theory of MPT, variance of the portfolio is expressed as:

$$V = Var[R_P] = \sum_{j=1}^{N}\sum_{i=1}^{N} xi\ xj\ Cij$$

This can be rewritten as:

$$V = \sum_{i=1}^{N} Vi\ xi^2 + \sum_{j=1}^{N}\sum_{\substack{i=1 \\ i\neq j}}^{N} xi\ xj\ Cij$$

In case of all independent assets, the covariance between them is zero and the formula for variance becomes:

$$V = \sum_{i=1}^{N} xi^2\ \ Vi$$

Now if there are N assets and equal amount is invested in each of the N assets, the proportion invested in each is 1/N. Hence, V = $\sum_{i=1}^{N}(\frac{1}{N})^2 Vi$ = (1/N) $\sum_{i=1}^{N}\left(\frac{1}{N}\right)Vi$

= $\frac{\overline{V}}{N}$

Where $\overline{V}$ represents the average variance of the stocks in the portfolio. As N gets larger and larger, the variance of the portfolio approaches zero. In other words, in the presence of enough independent assets, a lower variance i.e. a lower risk can be achieved.

[4]

ii) However, **in case of not so independent assets**, i.e. when the correlation coefficient and the covariance between assets is positive, the formula for the variance of the portfolio becomes:

V = $\sum_{i=1}^{N}(1/N)^2 Vi + \sum_{j=1}^{N}\sum_{\substack{i=1 \\ i\neq j}}^{N}(\frac{1}{N})(\frac{1}{N})Cij$

= (1/N) $\sum_{i=1}^{N}(1/N)Vi + \frac{(N-1)}{N}\sum_{j=1}^{N}\sum_{\substack{i=1 \\ i\neq j}}^{N}\frac{Cij}{N(N-1)}$

Replacing variances and covariances with their averages V and C,

V = $\frac{\overline{V}}{N} + \frac{N-1}{N}\overline{C}$

As N gets very large, the contribution to the portfolio variance of the variances of individual securities goes to zero. However, the contribution of the covariance terms approaches the average covariance as N gets large.

So the individual risk of securities can be diversified away but the contribution to the total risk caused by the covariance terms cannot be diversified away.

[3]

[7 Marks]

Solution 7:

i) Value of the incentive scheme

Considering the share options. The employee will receive 1,000 shares if the share price in 6 months' time is greater than:

Rs 10*1.1 =11

This is like having 1,000 call options on the share with a strike price of Rs 11, except that no payment is actually required. Using the Black-Scholes formula for one call option, (the Garman-Kohlhagen formula with *q* =0):

$$C_{[K=11]} = 10\,\Phi(d1) - 11\exp(-0.05*6/12)\,\Phi(d2)$$

But the employee will not need to pay the Rs 11 so the second term is not required. So,
the value of each share option is:

$$10\,\Phi(d1) = 10\,[\ln(S/K) + (r + \sigma^2/2)*(T-t)] / (\sigma\sqrt{(T-t)})$$

$10\,[\ln(1/1.1) + (0.05 + 0.25^2/2) * 0.5\,] / (0.25\sqrt{(0.5)})$

$10\,\Phi(-0.30935)$

$10 * (1-0.6215)$

$=3.785$

So, the share options for each manager are worth 1,000 *Rs 3.785 = Rs 3,785

Next, consider the Rs 5,000 cash incentive. The employees will receive this if the revenue from new connections are greater than:
100*1.2 = Rs 120 (assuming revenue is currently 100)

The value of a call option here would be:

$C_{[K=12]} = 100\Phi(d1) - 120\exp(-0.05*6/12)\,\Phi(d2)$

The second term here corresponds to the value of the strike price Rs 120 that would be paid if the revenue is greater than Rs 120 in 6 months' time.
The cash incentive is made in the same circumstances as this. However, the amount is Rs 5,000 rather than Rs 120.
So the value of the cash incentive is:

$5000\exp(-0.05*6/12)\,\Phi(d2) = 5000*0.97531\,[\ln(S/K) + (r - \sigma^2/2)*(T-t)\,] / (\sigma\sqrt{(T-t)})$

$= 5000*0.97531\,[\ln(10/12) + (0.05-0.25^2/2)*(0.5)\,] / (0.25\sqrt{(0.5)})$

$= 4876.55 * \Phi(-0.97833)$

$= 4876.55 * (1-0.8360)$

$= 799.75$

$= 800$

Finally, the total value of the incentive scheme is:
Rs 3,785 + Rs 800 = Rs 4,585

[6]

ii) Incentive for Employees to accept the offer?

- Both the Rs 5000 cash incentive and the share options do not give employees any incentive to help the company once the 6-month period is over.

- Employees may be able to sell their free shares and may have little interest in how the company subsequently performs.
- In addition, because the employees do not receive any incentive at all whether the share price increases by 9%, or decreases by 50% say as such, they may be tempted to undertake a riskier investment strategy that is against the performance of the shares.
- Any increase in revenue above 20% is not further rewarded, hence employees may be discouraged to boost the connections further [2]

iii) Improvement to the scheme

- Lock-in period: Restriction on selling the shares for a fixed time period, say 5 years, once awarded.
- Long term incentive plan: Imposing a condition that the shares will only be awarded provided the employee continues to work for the company for a fixed time period, 5 years say.
- Shares as against cash bonus: Instead of a cash bonus, the employees could be given the equivalent amount in more bonus shares, again with the restrictions mentioned above.
- The number of free shares issued could be made to depend more gradually on the company's share price performance, eg 10 free shares for every percentage point performance above a specified benchmark level.
- Employee Stock Option Scheme (ESOPs) may be given against any increment in the salary, if possible, so that their interests are aligned with the shareholders.

[2]

[10 Marks]

Solution 8:

i) Given a probability measure **P** and a history (filtration) of past events $\{F_t, t \leq s\}$, then the stochastic process $\{X_t, t \geq 0\}$ is a martingale if:

$E_P[X_t \mid F_s] = X_s$ for any $t \geq s$

In other words, the expected future value of the stochastic process X_t is its current value, i.e. it is driftless.

[1]

ii) We have:

$$\frac{\mu_f - r}{\sigma_f} = \lambda = \frac{\mu_g - r}{\sigma_g}$$

which implies

$\mu_f = \sigma_f \lambda + r$ & $\mu_g = \sigma_g \lambda + r$

Substituting $\lambda = \sigma_g$ gives

$\mu_f = \sigma_f \sigma_g + r$

$\mu_g = \sigma_g^2 + r$

Substituting these equations into the SDE s for f and g gives

$df = (\sigma_f \sigma_g + r) f dt + \sigma_f f dw$ ____(1)

$dg = (\sigma_g^2 + r) g dt + \sigma_g g dw$ ____(2)

We know that $\phi = \frac{f}{g}$

Therefore $ln\ \phi = ln\ (f/g) = lnf - lng$

Which implies $dln\ \phi = dln\ f - dlng$

Using Ito's Lemma on (1) and (2) for the functions ln f and ln g respectively, we get:

$dlnf = (\sigma_f \sigma_g + r - \sigma_f^2/2)\ dt + \sigma_f dw$ ____(1)
$dlng = (\sigma_g^2 + r - \sigma_g^2/2) dt + \sigma_g dw = (r + \sigma_g^2/2) dt + \sigma_g dw$ ____(2)

[Alternatively, we may use Ito on the original processes and then substitute for μ f and μ g to get the same equations]

$dln\ f = (\mu_f - \sigma_f^2/2)\ dt + \sigma_f dw$
$dln\ g = (\mu_g - \sigma_g^2/2) dt + \sigma_g dw$

So that, taking differences of the above differential equations,

$d(ln\ f - ln\ g) = (\sigma_f \sigma_g - \frac{\sigma_f^2}{2} - \frac{\sigma_g^2}{2})\ dt + (\sigma_f - \sigma_g) dw$

i.e. $d(ln(f/g)) = -(\frac{\sigma_f - \sigma_g}{2})^2 dt + (\sigma_f - \sigma_g) dw$

Using Ito s Lemma in reverse (comparing to formulae (1) and (2) above), we can write down the process for *f* / *g* that gives such a result for ln(*f* / *g*)

$d(f/g) = (\sigma_f - \sigma_g)\ (f/g) dw$

Alternatively

$dln\ \phi = dln\ f - dln\ g = [(\mu_f - \sigma_f^2/2) - (\mu_g - \sigma_g^2/2)]\ dt + (\sigma_f - \sigma_g) dw$

$= [\ (\mu_f - \mu_g) - \frac{1}{2}(\sigma_f^2 - \sigma_g^2)]\ dt + (\sigma_f - \sigma_g) dw$

$= [\ (\sigma_f \sigma_g + r - \sigma_g^2 + r) - \frac{1}{2}(\sigma_f^2 - \sigma_g^2)]\ dt + (\sigma_f - \sigma_g) dw$

$= [\sigma_g \sigma_f - \sigma_g^2 - \frac{1}{2}(\sigma_f^2 - \sigma_g^2)]\ dt + (\sigma_f - \sigma_g) dw$

$= -\frac{1}{2}(\sigma_f - \sigma_g)^2 dt + (\sigma_f - \sigma_g) dw$

According to Ito's lemma

$d\phi = [-\frac{1}{2}(\sigma_f - \sigma_g)^2 + \frac{1}{2}(\sigma_f - \sigma_g)^2]\ \phi\ dt + (\sigma_f - \sigma_g)\ \phi\ dw$

$= (\sigma_f - \sigma_g)\,\phi\,dw$

This is a driftless process and is hence a Martingale. [8]

iii) A world which consists of a security (or stochastic process) *g* whose volatility σ_g is equal to the market price of risk, then the world is said to be forward risk neutral with respect to *g.*

If *f* is any other security, then in the world that is forward risk neutral with respect to g, *f* / *g* is a Martingale. It follows that the current value of *f* / *g*, viz (f_0/g_0) is equal to the expected value at time zero of all future values $Eg[f_t/g_t]$, where expectations are carried out using the probability density function underlying *g*.
[2]

iv) We are given that

$df = \mu_f f dt + \sigma_f f dw$

Let the call option claim be *x*(*f*, *t*).
Stochastic process from Ito's Lemma is:

$dx = \frac{\partial x}{\partial f}df + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}(df)^2 + \frac{\partial x}{\partial t}dt$

$= \frac{\partial x}{\partial f}(\mu_f f dt + \sigma_f f dw) + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}(\sigma^2_f f^2)dt + \frac{\partial x}{\partial t}dt$

$= (\frac{\partial x}{\partial f}\mu_f f + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2_f f^2)dt + \frac{\partial x}{\partial f}\sigma_f f dw$ ________*(i)*

Using the replication argument, construct a portfolio π consisting of one unit of derivative and α units of stock

$\pi = x + \alpha f$

Over a small time interval,

$\Delta\pi = \Delta x + \alpha\,\Delta f$

hence, using the stochastic process above in its discrete version:

$\Delta\pi = (\frac{\partial x}{\partial f}\mu_f f + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2_f f^2)\Delta t + \frac{\partial x}{\partial f}\sigma_f f\Delta w) + \alpha\,(\mu_f f\Delta t + \sigma_f f\Delta w)$

$= (\frac{\partial x}{\partial f}\mu_f f + \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2_f f^{\,2} + \alpha\mu_f f)\Delta t + (\frac{\partial x}{\partial f}\sigma_f f + \alpha\,\sigma_f f)\Delta w$

Thus, if α is chosen to be $\alpha = -\frac{\partial x}{\partial f}$, then

$\Delta\pi = \Delta t(\frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2_f f^{\,2})$

Since the portfolio is riskless, it will earn the riskless rate of return, i.e.

$\Delta\pi = r\pi\Delta t.$

Thus,

$r\,(x - \frac{\partial x}{\partial f}f) = \frac{\partial x}{\partial t} + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2_f f^{\,2}$

i.e $rx = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial f}\, \mathrm{r}f + \frac{1}{2}\frac{\partial^2 x}{\partial f^2}\sigma^2{}_f f^2$ [6]

v) Boundary conditions

$x \geq 0$ for $0 \leq t \leq T$
$x = \max(f^2 - K, 0)$ at $t = T$
$x \geq f^2 - K$ for $0 \leq t \leq T$ (i.e. the American feature) [2]

vi) The differences in drift between F and G are not relevant, since drift has dropped out of the differential equation.

Since the volatility element is deterministic (i.e. not stochastic), the same differential equation of value is valid ...
... but with the previous volatility replaced by the dampened one.

In algebraic terms, this is:
$rx = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \mathrm{g}}\, \mathrm{rg} + \frac{1}{2}\frac{\partial^2 x}{\partial \mathrm{g}^2}\sigma^2{}_f \mathrm{g}^2 e^{-2\beta t}$ [3]

[22 Marks]

Solution 9:

i) In the binomial model it is assumed that:

- there are no trading costs or taxes
- there are no minimum or maximum units of trading
- stock and bonds can only be bought and sold at discrete times 1, 2, ...
- the principle of no arbitrage applies.

[2]

ii)

a) $n = t / \delta t$ [1]

b) Let *Xn* be the number of up jumps, *Yn* be the number of down jumps.

Then $Xn + Yn = n$,
so $Xn - Yn = 2Xn - n$. (*)

Now by simple multiplication, $S_t = S_0 \exp(\mu(n\delta t) + \sigma\sqrt{\delta t}\,(Xn - Yn))$
and the answer follows from using (*) and the answer to part (ii). [3]

c) *Xn* ~ Binomial with mean n/2 and variance n/4,
using the definition of an up and down jump with equal probability.

$E[\frac{2X_n - n}{\sqrt{n}}] = [\frac{2*(\frac{n}{2}) - n}{\sqrt{n}}] = 0$ and

Var $\left[\frac{2X_n-n}{\sqrt{n}}\right] = \frac{4*(\frac{n}{4})-0}{n}] = 1$

i,e $\left[\frac{2X_n-n}{\sqrt{n}}\right]$ has mean 0 and variance 1.

By the Central Limit Theorem, this variable converges to N(0, 1).
So as $\delta t \to 0$ and $n \to \infty$, the distribution of St approaches log-Normal, as log(St) is Normal with mean log($S0$) + μt and variance $\sigma^2 t$. [4]

d) $S_t = s\exp(\mu\delta t \pm \sigma\sqrt{\delta t})$

$= s\,(1 + (\mu\delta t \pm \sigma\sqrt{\delta t}) + \frac{(\mu\delta t \pm \sigma\sqrt{\delta t})^2}{2!} + \cdots$

Given a continuously compounded risk-free rate r, the risk-free up probability is

$$q = \frac{s\exp(r\delta t) - s_{down}}{s_{up} - s_{down}}$$

$$= \frac{[1 + r\delta t] - [1 + (\mu\delta t + \sigma\sqrt{\delta t}) - \frac{1}{2}\delta t(-2\mu\sigma\sqrt{\delta t} + \sigma^2\ )]}{2\sigma\sqrt{\delta t}}$$

$$= \frac{1}{2}\left(1 - \sqrt{\delta t}\left(\frac{\mu + \left(\frac{1}{2}\right)\sigma^2 - r}{\sigma}\right) + \ \mu\delta t\right)$$

to order δt , with the down probability $1 - q$.

[3]

[13 Marks]

****