# Institute of Actuaries of India

## Subject CT1 – Financial Mathematics

## September 2018 Examination

# INDICATIVE SOLUTION

**Introduction**

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

**Solution 1:**

**i)** Under interest rate swap one party agrees to pay floating rate and receive a fixed interest and the other party agrees to pay a fixed interest rate and receive a floating interest rate.

The fixed payments are at a constant rate for an agreed term and the floating payments will be linked to the level of short- term interest rate. [2]

**ii)**

- Shares are issued by commercial undertakings and other bodies.
- Shareholders receive dividends paid out of profits after all other obligations have been met.
- The return depends on the dividends received and any increase /decrease in the market price of the shares.
- Higher risk than corporate bonds issued by the same company.
- Lowest ranking form of finance issued by companies.
- No legal obligation to pay dividends.
- Generally low initial yield but dividends expected to grow over time.
- No fixed redemption time.
- Voting rights are attached.
- Marketability varies according to size of company

[3]

**iii)** If 1 unit is invested in the equity, it will generate expected dividends as follows:

$d(1 + g)$, $d(1 + g)^2$, $d(1 + g)^3$ at year 1, 2, ……. Where d is the dividend and g is the dividend growth rate.
Let i be the real return obtained and e be the inflation rate

So, the equation of value, allowing for inflation between when the equity is purchased and the receipt of each dividend, is:

$$1 = \frac{d(1+g)}{(1+e)}v + \frac{d(1+g)^2}{(1+e)}v^2 + \frac{d(1+g)^3}{(1+e)}v^2 + \ldots\ldots\ldots$$

$$= da_{\infty} \text{ where } j = \frac{(1+e)(1+i)}{(1+g)} - 1$$

$$1 = \frac{d(1+g)}{(1+e)(1+i)-(1+g)} \Rightarrow i = \frac{d(1+g)+g-e}{(1+e)} = 5.0196\%$$ [4]

**iv)** Option 1
Two consecutive 3 year for certificates of deposits:
$10000 * (1 + \left(\frac{0.05}{4}\right))^{12} * (1 + \left(\frac{0.05}{4}\right))^{12}$ = 13,473.51

Option 2
5+1 year for certificates of deposits

$10000 * (1 + \left(\frac{0.0575}{4}\right))^{20} * (1 + \left(\frac{0.04}{4}\right))^4 = 13{,}843.83$

So he will choose option 2 and accumulated value in such case will be INR 13,843.83. [3]

**[12 Marks]**

**Solution 2:**

**i)** $(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \ldots\ldots\ldots\ldots + nv^n$

$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \ldots\ldots\ldots\ldots + nv^{n-1}$

$i(Ia)_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|}$

$i(Ia)_{\overline{n}|} = 1 + v + v^2 + v^3 \ldots\ldots\ldots v^{n-1} - nv^n$

$(Ia)_{\overline{n}|} = (\ddot{a}_{\overline{n}|} - n * v^n) / i$ [3]

**ii)** The expected rate of interest over the ten year period is:

E(i) = 0.3* 7 + 0.5* 8 + 0.2* 10 = 8.1%

Single premium is X such that X* (1.081)^10 = 1,00,000

Single premium X =Rs 45,892.64 [2]

**iii)** Expected profit is :

45,892.64 * (0.3 * (1.07)^10 + 0.5 * (1.08)^10 + 0.2* (1.10)^10) -1,00,000

= 45892.64 * 2.188356 – 1,00,000

= 1,00,429.452- 1,00,000

**= INR 429.45** [2]

**[7 Marks]**

**Solution 3:**

**i)** Let X be the initial quarterly payment, then we have :

$58{,}50{,}000 = 4*X*a^{(4)}_{\overline{15}|} + 4*40{,}000 * v^5\, a^{(4)}_{\overline{10}|} + 4*40{,}000 * v^{10}\, a^{(4)}_{\overline{5}|}$

4X * 7.1101 + 1,60,000* 0.567426* 5.89848

+ 1,60,000 * 0.321973 * 3.76316 = 58,50,000

28.4404X +5,35,512.15 + 1,93,861.75 = 58,50,000

28.4404 X = 51,20,626.11

**X = Rs1,80,047.61** [4]

**ii)** The loan repaid in the third year:

The initial annual amount of installment is 4X i.e. 7,20,190.45

The loan outstanding after the payment of eighth installment (at the end of second year) is:

$58{,}50{,}000 * (1+i)^2 - 7{,}20{,}190.45 * s^{(4)}_{\overline{2}|}$ @ 12%

= 73,38,240 – 7,20,190.45 * 2.213148

= 73,38,240 – 15,93,888.05

= 57,44,351.95

And just after the twelfth installment is paid, the loan o/s is:

58,50,000 * $(1+i)^3$ – 7,20,190.45 * $s^{(4)}_{\overline{3}|}$ @ 12%

= 82,18,828.80 – 7,20,190.45 * 3.52266

= 82,18,828.80 – 25,36,986.09

= 56,81,842.71

The capital repaid in the third year is therefore:

57,44,351.95 - 56,81,842.71 = **INR 62,509.24**

<u>The loan repaid in the thirteenth year:</u>

The quarterly installment in the last five years is 1,80,047.61 + 2*40,000 = Rs 2,60,047.61

The loan o/s at the end of 12$^{th}$ year is :

2,60,047.61 * 4 * $a^{(4)}_{\overline{3}|}$

The loan o/s at the end of 13$^{th}$ year is :

2,60,047.61 * 4 * $a^{(4)}_{\overline{2}|}$

The capital repaid in the 13$^{th}$ year is

10,40,190.44 *( $a^{(4)}_{\overline{3}|} - a^{(4)}_{\overline{2}|}$ )
=10,40,190.44 * ( 2.50736 – 1.76431)
= **INR 7,72,913.51** [9]

**iii)** The loan o/s after the 33$^{rd}$ installment is paid :
10,40,190.44 * $a^{(4)}_{\overline{6.75}|}$ - 1,60,000* $a^{(4)}_{\overline{1.75}|}$
10,40,190.44 * 4.65119 – 160000 * 1.56501
48,38,123.37 – 2,50,401.60
= 45,87,721.77
Let the revised quarterly payment be Y, then we have:

$4 * Y\, a^{(4)}_{\overline{6.75}|} = 45,87,721.77$

$4Y * 4.65119 = 45,87,721.77$

Y = **INR 2,46,588.60** [5]

**[18 Marks]**

**Solution 4:**

**i)**

**a)** The TWRR for the fund:

$(19.36/10)^{(1/6)} - 1 = 11.64\%$ [1]

**b)** Yield obtained by the investor:

Let U(r) be the unit price on 1 April 2011 + r. The yield per annum

is found from the equation:

$500 * \sum(r=0 \text{ to } r=5)\, U(r) * (1+i)^{(6-r)} = 6 * 500 * U(6)$

$1000 * (1+i)^{6} + 1156 * (1+i)^5 + 1289 * (1+i)^4 + 1416 * (1+i)^3 +$

$1584 * (1+i)^2 + 1754 * (1+i) = 11616$

@ i= 11%, LHS = 11,610

@ i= 11.5%, LHS = 11,793

By interpolation, **i = 11.02%** [5]

**c)** Yield obtained i by Mr. Sam will satisfy for equation of value:

$25000 * \ddot{s}_{\overline{6}|} = 25000 * ((1/10) + (1/11.56) + (1/12.89)$

$+ (1/14.16) + (1/15.84) + (1/17.54)) * 19.36$ (where $\ddot{s}_{\overline{6}|}$ is calculated with i)

$\ddot{s}_{\overline{6}|} = 0.45485 * 19.36 = 8.8059$

@ i= 11% , LHS = 8.78

@ i= 11.5% , LHS = 8.93

By interpolation, **i= 11.08%** [5]

**ii)** The revised answer to (i)b)

500 * ∑(r=0 to r=5)((1.02 * U(r)) * $(1+ i)^{(6-r)}$ = 6 * 500 *0.98* U(6)

1000 * $(1+i)^6$ +1156 *$(1+i)^5$ + 1289 * $(1+i)^4$ + 1416 * $(1+ i)^3$ +

1584 *$(1+i)^2$ + 1754 * (1+i) = 11,160.47

@ i= 10%, LHS = 11,251

@ i= 9%, LHS = 10,902

**By interpolation, i= 9.74%**

The revised answer to (i)c)

25000 * $\ddot{s}_{\overline{6|}}$ = 25000*(1/1.02)* ((1/10) + ( 1/ 11.56) + ( 1/ 12.89)

+ (1/ 14.16)+ ( 1/ 15.84) + ( 1/ 17.54))* 19.36* 0.98

$\ddot{s}_{\overline{6|}}$ = 0.4548 *(1/1.02)* 19.36* 0.98= 8.46057

@ i= 10% , LHS = 8.4872

@ i= 9.5% , LHS = 8.343

By interpolation, **i= 9.91%** [7]

**[18 Marks]**

**Solution 5:** The price of Rs 100 stock is determined by the equation of value:

P(1) = 20 $a_{\overline{n|}}$ + 100 $v^n$ = 20 * $(1-v^n)$/i + 100 * $v^n$

= 20/i + (100 – 20/i) * $v^n$

When i=11%, the price is Rs 153

So: 153 = 20/ 0.11 + (100 – 20/ 0.11)* $1.11^{-n}$

Solving for n, gives n= 10 years

The price of Rs 100 stock, valued at 10% and allowing for the possibility

of default is:

P(2) = 20 ∑(t=1 to t=10) 0.95652^t / 1.10^t

+ 100 * 0.95652^t / 1.10^10

0.95652/1.10 = 1/1.15

P(2) = 20 * $a_{\overline{10|}}$ + 100 $v^{10}$ @ 15%

P(2) = 20* 5.0188 + 100 * 0.24718 = Rs 125.09 **[6 Marks]**

**Solution 6:**

Let the nominal amount of the security be 100
Hence the equation satisfying purchase price of tax-free investor is
132.55 = 6 * $a_n^{(2)}$ + C * $v^n$
The equation satisfying purchase price of taxed investor is
107.85 = 6 * (1-0.32) * $a_n^{(2)}$ + C * $v^n$
The value of income tax payment at 4% is

6 x 0.32 x $a_n^{(2)}$ = (132.55 – 107.85)

Hence $a_n$ = $\frac{i^{(2)}}{i}$ x $\frac{24.70}{1.92}$ = $\frac{0.039608}{0.04}$ x $\frac{24.70}{1.92}$ = 12.7488

Solving we get n= 18.16 or approximately 18 years (rounded to nearest integer)

The value of the interest payment for the Tax-free investor is

6 x $a_{18}^{(2)}$ = 6 x 12.7847 = 76.7079

The value of the capital repayment C x$v^{18}$ = 132.55 – 76.7079 = 55.8421
Hence C = 113.13 **[5 Marks]**

**Solution 7:**

**i)**

**a)** $(1+\frac{i^{(2)}}{2})^2 = (1+\frac{i^{\wedge}((12)}{12})^{12}$

$(1+\frac{i^{(2)}}{2}) = (1+\frac{0.07}{12})^6$

$i^{(2)} =$ 7.103% [2]

**b)** $(1-\frac{d^{(12)}}{12})^{12} = (1+\frac{i^{\wedge}((12)}{12})^{-12}$

$(1-\frac{d^{12}}{12})$ = $(1+\frac{0.07}{12})^{-1}$

$d^{(12)} =$ 6.959% [2]

**ii)** δ(t) = $\delta_0 + \frac{t}{m}(\delta_m - \delta_0)$ for 0 <= t <= m

δ(t)= $\delta_m$ for t>m

Required accumulated value

exp ($\int_0^n$ δ(t)dt

= Exp { $\int_0^m[\delta_0 + \frac{t}{m}(\delta_m - \delta_0)]$ dt + $\delta_m\ (n-m)$}

=exp { $[\delta_0\, t + \frac{t^2}{2m}(\delta_m - \delta_0)]_{t=0}^{t=m} + \delta_m\ (n-m)$}

= exp $[\delta_0\, m + \frac{m}{2}(\delta_m - \delta_0) + \delta_m\ (n-m)]$

= exp $[\frac{m}{2}(\delta_0 - \delta_m) + \delta_m\ n]$

= $[\exp(\delta_0 - \delta_m)]^{m/2}\,[exp(\delta_m)]^n$

= $(\frac{1.04}{1.03})^8\,(1.03)^{39}$ = 3.4215 [6]

**[10 Marks]**

**Solution 8:**

**i)** Purpose of models
Both types are used to project the accumulated value of flows of money.

Assumptions
Deterministic models assume that future rates of return are fixed. Stochastic models assume that future rates of return are random variables.

Results obtained
For a given assumed set of future rates of return, a deterministic model will give a single definite answer. For a given assumption about the statistical distribution of future rates of return, a stochastic model will give a statistical distribution describing a range of possible answers.

Risk and uncertainty
Stochastic interest rate models make allowance for uncertainty by enabling the probability that the actual value will lie in a given range to be calculated. Deterministic models make allowance for uncertainty by carrying out calculations based on different sets of assumptions (eg by including

contingency margins in the assumptions). [3]

**ii)**

**a)** (1+ $i_t$ ) ~ Log N (μ , $\sigma^2$ )

E (1+ $i_t$ ) = $e^{(\mu+\sigma^2/2)}$ =1.05

$(\mu + \sigma^2/2)$ = 0.048790

Var (1+ $i_t$) = $e^{(\mu+\sigma^2/2)}(e^{\sigma^2} - 1)$

= $(1.05)^2$ $(e^{\sigma^2} - 1)$ = $0.11^2$

$e^{\sigma^2}$ = 1.010975

⇨ $\sigma^2$ = 0.010915

⇨ $\mu = 0.04333$ [3]

**b)** Pr [ (1.04) < (1+$i_t$) < 1.07)

=ϕ[$\frac{\log(1.07)-\mu}{\sigma}$ ] - ϕ[$\frac{\log(1.04)-\mu}{\sigma}$ ]

= ϕ[0.233] - ϕ[-0.039]

= 0.59211- (1-0.51555)

= 0.10766 [3]

**[9 Marks]**

**Solution 9:**

**i)** The instantaneous forward rate may broadly be thought of as the forward force of interest applying in the instant of time t to t + Δt

Expression:

$F_t = \lim_{r-0} F_{t,r}$ [1.5]

**ii)** The liquidity preference theory states that, as a general rule, investors prefer to hold stocks that they are not locked into for a long time period.

They are therefore prepared to pay more for shorter, more liquid stocks.

This implies that the yield obtainable on these shorter stocks will be lower. Hence the yield curve will normally have an upward slope. [1.5]

**[3 Marks]**

**Solution 10:**

**i)** PV of outgoings = 18,00,000 x $\ddot{a}_3$ at 7%

= 18,00,000 x 2.80802

= 50,54,436

PV of incoming payments = 2,50,000 $\int_0^{25}(1.06)^t\,(1.07)^{-t}$ dt

This is equivalent to present value of annuity with $v^t$ =$(\frac{1.06}{1.07})^t$

= $0.990654^t$ *and i*= 0.943396%

Hence, PV of incoming payments = 2,50,000 $\int_0^{25}(1.06)^t\,(1.07)^{-t}$ dt

= 2,50,000 x $\bar{a}_{25}$ at *i*= 0.943396%

= 2,50,000 x $\left(\frac{1-0.990654^{25}}{\delta}\right)$ where δ = 0.00938974

= 55,70,720

NPV = 5,16,284 [6]

**ii)** Need to find t such that

50,54,436 = 2,50,000 x $\left(\frac{1-0.990654^{t}}{0.00938974}\right)$

0.189839 = ( 1- 0.990654 ^t)

i.e. 0.810161 = 0.990654^t

i.e. t = 22.42 [2]

**[8 Marks]**

**Solution 11:**

**i)** 20,00,000 = 8% x[2,50,00,000 + 0.5 * (X- 22,00,000-7,50,000)]

= 18,82,000 +0.04 X

X = 29,50,000

Assets at the end of the year

= 2,50,00,000+29,50,000 +20,00,000-22,00,000-7,50,000

=2,70,00,000 [2]

**ii)** Hence effective yield is

2,70,00,000 =2,50,00,000 * (1+i) +( 29,50,000- 22,00,000-7,50,000)*$(1+i)^{0.5}$

2,70,00,000 =2,50,00,000 * (1+i) +0*$(1+i)^{0.5}$

(1+i) = 1.08 => i = 8% [2]

**[4 Marks]**

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