Institute of Actuaries of India

Subject CT8 – Financial Economics

September 2017 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:-

i) a) **Value at Risk**

Value at Risk generalizes the underperforming by providing a statistical measure of downside risk.

For a continuous random variable, Value at Risk can be determined as:

VaR (X) = -t where P (X<t) = p

VaR represents the maximum potential loss on a portfolio over a given future time period with a given degree of confidence, where the latter is normally expressed as 1-p.
For a discrete random variable, VaR is defined as:

VaR(X) = -t where t = max{x : P(X < x) ≤ p}

(distribute 0.5 marks each for definition , notation for continuous random variable and notation for discrete random variable)

b) **Tail value at Risk**

This risk measure can be expressed as expected shortfall below a certain level.

For continuous random variable, the expected shortfall is given by:

Expected shortfall = E[max(L-X, 0)] = $\int_{-\infty}^{L}(L-x)f(x)dx$

where L is the chosen benchmark level

For a discrete random variable, the expected shortfall is given by:

Expected shortfall = E[max(L-X, 0)] = $\sum_{x<L}(L-x)\ P(X=x)$

If L is chosen to be a particular percentile point on the distribution, then the risk measure is known as the Tail Value at Risk.

[3]

ii) Risk Measure & Utility Function

An investor using a particular risk measure will base his decisions on a consideration of the available combinations of risk and return. Given knowledge of how this trade-off is made it is possible, in principle, to construct the investor's underlying utility function. Conversely, given a particular utility function, the appropriate risk measure can be determined.

For example, if an investor has a quadratic utility function, the function to be maximized in applying the expected utility theorem will involve a linear combination of the first two moments of the distribution of return. Thus variance of return is appropriate in this case.

If expected return and semi-variance below the expected return are used as the basis of investment decisions, it can be shown that this is equivalent to a utility function that is quadratic below the expected return and linear above.

Thus, this is equivalent to the investor being risk-averse below the expected return and risk-neutral for investment return levels above the expected return. Hence, no weighing is given to variability of investment returns above the expected return.

Use of a short fall risk measure corresponds to a utility function that has a discontinuity at the minimum required return. This therefore corresponds to a state-dependent utility functions. [2]

iii) R = 250,000- 100,000N

a) *N* has a Normal [1,1] distribution, so *R* has a Normal distribution with mean 150,000 and variance $100{,}000^2$, *ie* $R \sim N[150000, 10^{10}]$.
So, $Var(R) = 100{,}000^2 \, Var(N) = 10^{10}$

[1]

b) Any normal distribution is symmetrical about its mean, so that the downside semi-variance of return is equal to half of the variance, *ie* $5\text{x}10^9$,

[1]

c) N is a Normal [1, 1] i.e (N-1) is Normal [0, 1]
P(R < 50,000) = P(2,50,000-100,000N < 50,000)
= P(N > 2) = P(N-1 > 1)= 1 − Φ(1) = 1 − .8413 = .1587

[2]

d) If $VaR_{5\%}(R) = t$ then $P(R \leq -t) = 0.05$,
therefore P(250,000 − 100,000N ≤ -t) = P(N > 2.5 +(t/100,000)) = 5%,

hence (since N − 1 is a standard normal random variable)

Φ(1.5 + (t/100,000)) = .95, so t = 100,000(1.6449-1.5) = Rs. 14,490.

[3]

iv) Tail Value at Risk

The VaR is 14,490. So, conditional TVaR is:

$$\frac{1}{.05}\int_{-\infty}^{-14{,}490}(-14{,}490 - x)f(x)dx$$

where *f* (*x*) is the *pdf* of R which has *N*(150,000, 10^{10}) distribution.
Conditional TVaR = $\frac{-14{,}490}{.05}\int_{-\infty}^{-14{,}490} f(x)dx - \frac{1}{.05}\int_{-\infty}^{-14{,}490} x f(x)dx$

[3]
[15 Marks]

Solution 2:

i)

Asset	Expected Return	Standard Deviation
1	.06	.10

2	.08	.15
3	.10	.20

Correlation matrix is

$$\begin{bmatrix} 1 & .5 & .5 \\ .5 & 1 & .5 \\ .5 & .5 & 1 \end{bmatrix}$$

Variance and covariance matrix can be determined as

$$\begin{bmatrix} .01 & .0075 & .01 \\ .0075 & .0225 & .015 \\ .01 & .015 & .04 \end{bmatrix}$$

Where
$C_{ij} = \rho_{ij}\sigma_i\sigma_j$

Lagrangian function satisfies

$W= \sum_{j=1}^{3} \sum_{i=1}^{3} x_i x_j C_{ij} - \lambda(\sum_{i=1}^{3} x_i E_i - E) - \mu(\sum_{i=1}^{3} x_i - 1)$

$= (.01x_1^2 + .0225\, x_2^2 + .04\, x_3^2) + 2*(.0075\, x_1 x_2 + .015\, x_2 x_3 + .01 x_3 x_1)$

$- \lambda(.06\, x_1 + .08 x_2 + .10 x_3 - .09) - \mu\,(x_1 + x_2 + x_3 - 1)$

Where λ and μ are Lagrangian multipliers, x_i are the proportion of assets, E_i is expected return on each asset and E is expected return on the portfolio

[3]

ii) Equating partial derivative of W w.r.t. x_i to 0, we get

$\frac{\partial W}{\partial x_1} = .02\, x_1 + .015 x_2 + .02\, x_3 - .06\lambda - \mu = 0$

$\Rightarrow\ .06\lambda + \mu = .02\, x_1 + .015 x_2 + .02\, x_3$ ----------- (A)

$\frac{\partial W}{\partial x_2} = .045 x_2 + .015\, x_1 + .03\, x_3 - .08\lambda - \mu = 0$

$\Rightarrow\ .08\lambda + \mu = .015\, x_1 + .045 x_2 + .03\, x_3$ ----------- (B)

$\frac{\partial W}{\partial x_3} = .08\, x_3 + .03 x_2 + .02\, x_1 - .1\lambda - \mu = 0$

$\Rightarrow\ .1\lambda + \mu = .02 x_1 + .03 x_2 + .08 x_3$ ----------- (C)

$\frac{\partial W}{\partial \lambda} = .06 x_1 + .08 x_2 + .10 x_3 - .09 = 0$----------- (D)

$\frac{\partial W}{\partial \mu}$ = x_1+x_2 +x_3- 1=0----------- (E)

[4]

iii) Corner portfolio where *x*1 = 0. Equations become
x_2 +x_3=1 => x_3=1- x_2

.06λ+ μ = .015x_2 +.02 x_3 = .015x_2 +.02(1- x_2)---------- (A)
.06λ+ μ = .02-.005 x_2 ---------- (I)

.08λ+ μ = .045x_2 +.03 x_3= .045x_2 +.03(1- x_2)----------- (B)

.08λ+ μ = .03+.015x_2 ---------- (II)

.1λ+ μ = .03x_2 +.08(1-x_2) ----------- (C)

.1λ+ μ = .08 -.05x_2---------- (III)

Solving (I), (II) and (III) we get
x_2 =.4706 i.e. 47.06%
x_3 =.5294 i.e. 52.94%

[5]
[12 Marks]

Solution 3:

i) Within the context of CAPM, the market price of risk is defined as:

Market- price of risk = $\frac{(E_M - r)}{\sigma_M}$

Where
E_M = the expected return on market portfolio
r = the risk free rate of return
σ_M = the standard deviation of market portfolio returns

It is the additional expected return that the market requires in order to accept an additional unit of risk, as measured by the portfolio standard deviation of return.

It is equal to the gradient of the capital market line in E – σ space

[2]

ii) **a)** E_P = 18%
σ^2_M = 4%% => σ_M = 2%
r = 4%
E_M = 12%
$E_p - r = \sigma_p \frac{(E_M - r)}{\sigma_M}$

$18 - 4 = \sigma_p \frac{(12 - 4)}{2}$

⇨ $\sigma_p = 3.5\%$

[2]

b)

The efficient portfolio is a mix of the market portfolio and the risk-free asset. If the weights (which sum to 1) are x_M and x_0, then the expected return is

$x_M E_M + x_0 r = Ep$

$x_M * 12 + (1- x_M)*4 = 18$

⇨ $x_M = 1.75$

⇨ $x_0 = -0.75$

Thus the efficient portfolio has Rs. 2,100,000 in the market portfolio and is short Rs. 900,000 in cash. [3]

[7 Marks]

Solution 4

i) The assumptions underlying Black Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian Motion.
   i.e. the share price changes *continuously* through time according to the stochastic differential equation:
   $dS_t = S_t(\mu dt + \sigma dZ_t)$

2. There are no risk-free arbitrage opportunities.
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.
4. Unlimited short selling (that is, negative holdings) is allowed.
5. There are no taxes or transaction costs.
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

[2]

ii) How realistic are the assumptions?

1. Share prices can jump. This invalidates assumption 1 since geometric Brownian motion has continuous sample paths. However, hedging strategies can still be constructed which substantially reduce the level of risk.
2. The risk-free rate of interest does vary and in an unpredictable way. However, over the short term of a typical derivative the assumption of a constant risk-free rate of interest is not far from reality.
3. Unlimited short selling may not be allowed except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling.
4. Shares can normally only be dealt in integer multiples of one unit, not continuously and dealings attract transaction costs: invalidating assumptions 4, 5 and 6.
5. Distributions of share returns tend to have fatter tails than suggested by the log-normal model, invalidating assumption 1.

[2]

iii) Lower and upper bounds for European Call

Lower Bund of European call:

Consider a portfolio consisting of European call on a non-dividend paying share and a sum of money equal to K$e^{-r(T-t)}$

At time T, the portfolio value is equal to the value of the underlying share, if the share price is greater than K at T.

If the share price is less than K, then the value is K which is the accumulated value of the cash i.e. the payoff is greater than the value of the share.

Hence the lower bound is determined from c_t + $Ke^{-r(T-t)}$ >=S_t

Or c_t >=S_t-$Ke^{-r(T-t)}$

Upper Bound of European call:

The payoff is Max {0, S_T – K} which is always less than the value of the share at T.

Hence $c_t \leq S_t$

$S_t \geq c_t \geq$ max {S_t- $Ke^{-r(T-t)}$,0}

$60 \geq c_t \geq$ max {60- $50e^{-.03*3}$,0}

$60 \geq c_t \geq 14.30$

[3]

iv) Additional information is

Volatility: 25% pa, Vega: 29

S_0 = 60
r = 3% continuously compounded
K = 50
T-t= 3 year

$$d_1 = \frac{\ln\left(\frac{S0}{K}\right) + (r + 1/2\sigma^2)(\mathrm{T-t})}{\sigma\sqrt{(\mathrm{T-t})}}$$

$$= \frac{\ln\left(\frac{60}{50}\right) + (.03 + \frac{1}{2} * .25^2)(3)}{.25\sqrt{(3)}}$$

$$= \frac{.36607}{.43301}$$

= .8454

$d_2 = d_1 - \sigma\sqrt{(\mathrm{T}-\mathrm{t})}$ = 0.4124

Now from table $\Phi(d_1)$ = 0.79955 (approximately)

and
$\Phi(d_2)$= 0.65910

Garman- Kohlhagen formula for price of call option

$c_t = S_0 e^{-q(T-t)} \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2)$

For non- dividend paying share
c_t= 60*.79955 – 50*$e^{-.03*3)}$ *.65910
= 17.85

We see that 60 ≥ c_t ≥ 14.30
hence the boundary condition is satisfied.

[3]

v) Using the Taylor's approximation

$f(S, \sigma+\delta\sigma) = f(S, \sigma) + \delta \frac{\partial f}{\partial \sigma}$

= f (S, σ) + δ *ν (ν i.e. vega)

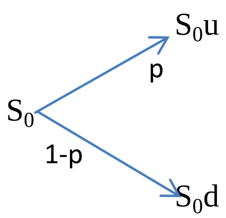
= 17.85 + (.27-.25)*29

= 18.43

[2]
[12 Marks]

Solution 5:

i) Setting up the commodity tree using u for up move and d for down move, p is up-step probability:



Where p is the up probability and (1-p) the down probability.

Then $E(C_t) = S_0[pu+(1-p)d]$, and

$Var(C_t) = E(C_t^2) - E(C_t)^2$

$= S_0^2 [pu^2+(1-p)d^2] - S_0^2[pu+(1-p)d]^2$

$= S_0^2 [pu^2+(1-p)d^2-(pu+(1-p)d)^2]$

$= S_0^2\, [p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)]$ $(\because d = 1/u)$

$= S_0^2\, p(1-p)(u-d)^2$

Equating moments:

$S_0 e^{rt} = S_0[pu+(1-p)d]$ _______________(A)

And $\sigma^2 S_0^2 t = S_0^2\, p(1-p)(u-d)^2$ _______________(B)

From (A) we get

$p = \frac{e^{rt}-d}{u-d}$ __________________________(C)

Substituting p into equation (B), we get

$\sigma^2 t = \frac{e^{rt}-d}{u-d}\left(1-\frac{e^{rt}-d}{u-d}\right)(u-d)^2$

$= -(e^{rt}-d)(e^{rt}-u) = (u+d)\, e^{rt} - (1+e^{2rt})$

Putting d = 1/u, and multiplying through by u we get

$u^2 e^{rt} - u\,(1+e^{2rt}+\sigma^2 t) + e^{rt} = 0$

This is a quadratic in u which can be solved in the usual way.

[3]

ii)

a) $\sigma = 0.15$, $t = 0.25 \Rightarrow u = \exp(.15 * \sqrt{.25}) = \exp(.075) = 1.077884$, $d = 1/u = .92774$

The tree is

t=0	t=.25	t=.5	t=.75	
			100.186	Node A
		92.947		
	86.231		86.232	Node B
80		80.001		
	74.22		74.22	Node C
		68.857		
			63.882	Node D

[3]

b) r=0, we have $p = \frac{e^{rt}-d}{u-d} = \frac{(1-.927744)}{(1.077884-.927744)} = .48126$

Discounting back the final payoff at t=.75 to t=0 along the tree using p and (1-p), we get

t=0	t=.25	t=.5	t=.75	
			20.186	Node A
		12.948		
	7.787		6.232	Node B
4.496		2.999		
	1.443		0	Node C
		0		
			0	Node D

Hence value of the call option is 4.496.

[2]

c) The lookback call pays the difference between the minimum value and the final value.

Notate paths by U for up and D for down, in order

We get the payoffs

UUU	(100.186 - 80)= 20.186	Node A
UDU	(86.232-80)= 6.232	Node B
UUD	(86.232- 80) =6.232	Node B
UDD	(74.22-74.22) = 0	Node C
DUU	(86.232- 74.22) = 12.012	Node B
DUD	(74.22-74.22) = 0	Node C
DDU	(74.22-68.857) = 5.363	Node C
DDD	(63.882-63.882)= 0	Node D

The lookback payoffs are, for each successful path (i.e. with a non- zero result)

Probabilities of arriving at each node are:

Node A= p^3 = .11147

Node B= $p^2(1-p)$ = .12015

Node C= $p(1-p)^2$ = .12950

Node D= $p(1-p)^3$ = .13959

Hence the tree value of lookback option is:

(.11147*20.186)+ (.12015*[6.232+ 6.232+ 12.012]) + (.12950*5.363)

= 5.8854

[5]

[13 Marks]

Solution 6:

i) Suppose that X_t is a martingale with respect to a measure P i.e

For any t<s Ep [Xs I Ft] = Xt

Suppose there is Y_t which is also another martingale with respect to P.

The martingale representation theorem states that there exists a unique previsible process Φt such that (in continuous time):

$Y_t = Y_0 + \int_0^t \Phi s \, dXs$

$dY_t = \Phi_t dX_t$

if and only if there is no other measure equivalent to P under which is X_t is a martingale [2]

ii)

To establish the derivative pricing formula using the martingale approach consists of 5 steps

Step 1: Find the unique martingale measure Q under which $D_t = e^{-rt} S_t$ is a martingale

Step 2: Let $Vt = e^{-r(T-t)} E_Q [X / Ft]$ where X is the derivative payoff at the time T. this is proposed as the fair price of the derivative at time t

Step 3 : $e^{-rt} E_Q [X / Ft] = e^{-rt} V_t$ This is martingale under Q

Step 4 By the martingale representation theorem, there exists a previsible process Φt

Such that $dE_t = \Phi_t \, dD_t$

Step 5 Let $\psi_t = Et - \Phi_t D_t$

And at time t hold the portfolio consisting of

- Φ_t units of tradable of S_t
- ψ_t units of cash account

At time t the value of this portfolio is equal to Vt. Also $V_T = X$

Therefore the hedging strategy (Φ_t, ψ_t) is replicating and so V_t is the fair price at time t [5]

iii) In PDE approach we have to guess the solution whereas the martingale approach we don't
Martingale approach provides an expectation that can be evaluated explicitly in some cases
The martingale approach can be applied to any F_T – measurable derivative payment, whereas the PDE approach cannot always
However the PDE approach is much quicker and easier to construct and more easily understood [2]

[9 Marks]

Solution 7:

i)

Under the Merton model, the value of debt is

min(F(4),120) = 120 - max(120 - F(4),0)

= F(4) – max(F(4) - 120, 0),

where F(t) is the gross value of the company at time t.

Thus the value at time 0 is

$e^{(-4r)}$E[min(F(4),120)]

= $e^{(-4r)}$ E[F(4) - max(F(4) - 120,0)],

[3]

ii) The bond price is 120 * $e^{-4(r+.04)}$ = INR 87.13 cr.

[1]

iii) The call price is 180 – 87.13 = 92.8621

with T = 4, r = 0.04, S0 = 180, K = 120.

This leads to an estimated volatility of 40%.

[3]

iv) Q(default) = Q(F(4) < 120)

= 1 - Φ(d2) = 1 - Φ(0.306831)
= 1 - 0.620514
= 0.379486

= 37.9%

[3]

v) Let Q(1) be the risk neutral probability that a corporate bond will default between time zero and 1 year.
Assuming there is no recovery in the event of default, probability is Q(1) that the bond will be worth 0 at maturity and probability is 1-Q(1) that it will be worth Rs. 100(principal amount).
The expected value of the bond is therefore {Q(1)*0+[1-Q(1)]*100}*exp(-rf)
where r_f is the 1 year risk free zero rate, r is the yield on the bond.
Then 100.exp(-r) =100[1-Q(1)]exp(-r_f) i.e. Q(1) = 1- exp(-r+ r_f)
Hence for investment grade,

Q(1) = 1-exp(-.012) = 1.19%.
For Junk grade
Q(1) = 1-exp(-.019) = 1.88% [3]

vi) The ratings transition matrix will be (the sum of transition probabilities from one state to the possible states is 1)

State	Investment	Junk	Default
Investment	0.9	0.0881	0.0119
Junk	0.1812	0.8	0.0188
Default	0	0	1

[2]

[15 Marks]

Solution 8:

i) Note that $dX_t = \frac{\partial f}{\partial t}(\mathrm{x}e^{at})dt + d[\int_0^t be^{a(t-s)}dW_s]$

Let $f(s,t) = be^{a(t-s)}$

So that $\frac{\partial}{\partial t}f(s,t) = bae^{a(t-s)}$, and $f(t,t) = b$

Then, by the stated result,

$d[\int_0^t be^{a(t-s)}dW_s] = dt\int_0^t bae^{a(t-s)}dW_s + bdW_t$

Also $\frac{\partial f}{\partial t}(\mathrm{x}e^{at}) = \mathrm{ax}e^{at}$

So, $dX_t = \mathrm{ax}e^{at}dt + dt\int_0^t bae^{a(t-s)}dW_s + bdW_t = aX_t dt + bdW_t$

[4]

ii) $X_t = aX_t dt + bdW_t = (-a)(0 - X_t)dt + bdW_t$

Thus, X_t will follow a Vasicek model if (-a) > 0 , or a < 0.
Comparing with the SDE of a Vasicek model, we know that X_t will have mean reversion to 0.

The mean and variance of the process can be derived as follows:

$X_t = e^{at}x + b\int_0^t e^{a(t-s)}dW_s$

For the first term:

E$[e^{at}x]$ = $e^{at}x$ and V$[e^{at}x]$ = 0

Considering the second term $b\int_0^t e^{a(t-s)}dW_s$,

Note that: $\int_0^t f(s)dB_s \sim N(0, \int_0^t f^2(s)ds)$

[Proof:

This follows from the fact that the MGF of a normal distribution $Y \sim N(\mu, \sigma^2)$ *is* $E(e^{qY}) = e^{(q\mu + q^2\sigma^2/2)}$

Let $M_b = e^{q\int_0^t f(s)dB_s - (\frac{1}{2})\int_0^t q^2 f^2(s)dB_s}$

M_b *is a martingale and hence* $E(M_b) = M_0 = 1$

So $E(e^{q\int_0^t f(s)dB_s - 1/2\int_0^t q^2 f^2(s)dB_s}) = 1$

Or, $E(e^{q\int_0^t f(s)dB_s}) = e^{(1/2)\int_0^t q^2 f^2(s)dB_s}$ *which is the MGF of* $\int_0^t f(s)dB_s$ *following* $N(0, \int_0^t f^2(s)ds)$ *]*

Applying this result on $f(s) = be^{a(t-s)}$, we get

$E(X_t) = e^{at}x + 0 = e^{at}x \to 0$ as $t \to \infty$, if $a < 0$

$V(X_t) = b^2\int_0^t e^{2a(t-s)}\,ds = b^2 e^{2at}\left(-\frac{1}{2a}\right)e^{-2as}]_0^t = \frac{b^2}{2a}e^{2at}(1 - e^{-2at})$

$= \frac{b^2}{2a}(e^{2at} - 1) \to (-\frac{b^2}{2a})$ as $t \to \infty$, if $a < 0$ [6]

iii) $X_t = e^{at}x + b\int_0^t e^{a(t-s)}dW_s$

$= e^{at}x + be^{at}\int_0^t e^{-as}dW_s$

$= e^{at}x + e^{a(t-t_1)}be^{at_1}\int_0^{t_1} e^{-as}dW_s + be^{at}\int_{t_1}^t e^{-as}dW_s$

$= e^{a(t-t_1)}X_{t_1} + be^{at}\int_{t_1}^t e^{-as}dW_s$

This shows $E(X_t | X_{t_1}) = e^{a(t-t_1)}X_{t_1}$ i.e. it is not equal to X_{t_1} which is the condition for the process to be a martingale. Hence the process is not a martingale. [4]

iv) Yes it is a Markov process as $f(X_t | X_S, s \le t_1) = f(X_t | X_{t_1})$

Note that

$X_t = e^{a(t-t_1)}X_{t1} + be^{at}\int_{t_1}^t e^{-as}dW_s$

$= g_1(t, X_{t1}) + g_2(s, X_s\,; t_1 \le s \le t)$

Hence, $f(X_t | X_s, s \le t_1) = f(X_t | X_{t1})$

i.e. distribution of $X_t | X_{t1}$ depends only on X_{t1}

[3]

[17 Marks]

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