Institute of Actuaries of India

Subject CT1 – Financial Mathematics

September 2017 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) The purchase price:

$1,20,000 * a^{(4)}_{\overline{5|}}$ @ 12% $= 1,20,000 * (i/i^{(4)}) * a_{\overline{5|}}$

$= 1,20,000 * (0.12/0.114949)*3.6048$

= INR 4, 51,581

[2]

ii) Taking half year as unit of time and 6% as the ROI.

$60,000 * a^{(2)}_{\overline{10|}}$ @ 6% $= 60,000 * (i/i^{(2)}) * a_{\overline{10|}}$

$= 60,000 * (0.06/0.05912)*7.36$

= INR 4, 48,127

[2]

iii) Taking quarter as unit of time and 3% as the ROI.

$30,000 * a_{\overline{20|}}$ @ 3% $= 30,000 * a_{\overline{20|}}$

$= 30,000 * 14.877747$

= INR 4, 46,324 [2]

[6 Marks]

Solution 2:

- Both preference shares and ordinary shares raise money for the issuing company.
- Preference shares are less common than ordinary shares. Assuming that the company makes sufficient profits, they offer a fixed stream of investment income.
- The crucial difference between preference shares and ordinary shares is that preference share dividends are limited to a set amount, which is almost always Paid.
- Preference shareholders rank above ordinary shareholders (both for dividends and, usually, on winding up), and only get voting rights if dividends are unpaid or if there is a matter which directly affects the rights of preference shareholders.
- Preference dividends, like ordinary dividends, are only paid at the directors' discretion, but no ordinary dividend can be paid if there are any outstanding preference dividends.
- For all investors, the expected return on preference shares is likely to be lower than on ordinary shares because the risk of holding preference shares is lower: preference shares rank higher on a winding-up, and the level of income payments is more certain.

[3 Marks]

Solution 3:

i) $v(t) = \exp(-\int_0^t \delta(s)\, ds)$

$v(t) = \exp(-\int_0^t 0.07\, ds) = \exp(-0.07t)$ for $0 \le t < 4$

$v(t) = \exp(-\int_0^4 0.07\, ds - \int_4^t 0.06\, ds) = \exp(-0.04 - 0.06t)$ for $4 \le t < 8$

$v(t) = \exp(-\int_0^4 0.07\, ds - \int_4^8 0.06\, ds - \int_8^t 0.05\, ds) = \exp(-0.08 - 0.04 - 0.05t)$

$v(t) = \exp(-0.12 - 0.05t)$ for $8 \le t$

[5]

ii) Accumulated value = 5000* v(3)/v(15)

= 5000* exp(-0.21)/ exp(-0.12-0.75)

= 5000 * exp(0.66)

= INR 9673.96 [2]

iii) Let $\delta(t)$ be the constant force of interest

$5000 * \exp(12\delta) = 5000 * \exp(0.66)$
$12\delta = 0.66$

$\delta = 0.055$ [2]

iv) $1/v(20) = 1/\exp(-0.12-1) = 3.06485 = (1+i)^{20}$

Where i is the effective annual rate

So i= 0.0576 [2]

v) Present value

1000*(1+v(1)+v(2) +……….v(9))

1000* (1+e(-0.07)+ e(-0.14)+ e(-0.21)+ e(-0.28)+ e(-0.34)+ e(-0.40)+ e(-0.46)+ e(-0.52)+ e(-0.57))

= INR 7,541.54 [4]

[15 Marks]

Solution 4:

i) In 10 years' time single premium P is:

P $= 12{,}000\left(a_{1}^{(12)} + 1.03a_{1}^{(12)}v + (1.03)^2 a_{1}^{(12)}v^2 + \cdots .. + (1.03)^{14} a_{1}^{(12)} v^{14}\right)$

$=12{,}000\, a_{1}^{(12)} \left(1 + \frac{1.03}{1.06} + \left(\frac{1.03}{1.06}\right)^2 + \cdots \ldots + \left(\frac{1.03}{1.06}\right)^{14}\right)$

$=12{,}000\, a_{1}^{(12)} \left(\frac{1-\left(\frac{1.03}{1.06}\right)^{15}}{1-\frac{1.03}{1.06}}\right)$

= 12,000 x 0.969067 x $\frac{0.3499146}{0.0283019}$

=1, 43,774.45

[5]

ii) $E(1+i_t) = 1.06 = e^{\mu+\frac{\sigma^2}{2}}$

$\text{Var}\,(1+i_t) = (0.15)^2 = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

Then $\sigma^2 = 0.01982706$

And μ = 0.04835538

s_{10} ~ LN(10*0.04835538, 10*0.01982706)

Let X be the amount to be invested at time 0

Pr (X s_{10} ≥ 1,43,774.45) =0.98

Pr (X s_{10} ≥ 1,43,774.45) =0.98

$1- \ Z\left(\frac{\ln\left(\frac{143774.45}{x}\right)-10\mu}{\sqrt{10\sigma^2}}\right) = 0.02$

$\text{Ln}\left(\frac{143774.45}{x}\right) = -2.0537 \times \sqrt{0.1982706} + 0.4835538$

X = 2,21,219.41

[7]

iii)

The initial investment needs to be substantially higher than the single premium required in 10 years' time to have a 98% probability of accumulating to the single premium.

The result is explained by the fact that the variance of the interest rate is so high relative to the mean. There is therefore a significant risk that the investment will decrease in value over the next 10 years.

[2]

[14 Marks]

Solution 5:

i) Discounted Mean Term =

$\frac{10\,v^{10}+11\,v^{11}+\cdots\ldots\ldots+20v^{20}}{v^{10}+v^{11}+v^{12}+\cdots\ldots+v^{20}}$

$= \frac{9\,a_{11}+(Ia)_{11}}{a_{11}}$ at 6%

$= 9 + \frac{(Ia)_{11}}{a_{11}}$

$= 9 + \frac{42.7571}{7.8869}$

= 14.42 [3]

ii) First condition PV (assets) = PV (Liabilities)

⇨ X v^{10} + Y v^{20} = $v^9\ a_{11}$ at 6%

⇨ X *0.55839 + Y * 0.31180 = 0.59190 * 7.8869 = 4.668256

Second condition is DMT assets = DMT liabilities

⇨ $\frac{X*10v^{10} + Y*20\,v^{20}}{X\,v^{10} + Y\,v^{20}}$ = 14.42

⇨ X* 5.5839 + Y * 6.236 = 14.42 * ($X\,v^{10} + Y\,v^{20}$)

⇨ X* 5.5839 + Y * 6.236 = 14.42 *4.668256 = 67.3163

Solving equations

Y =6.6176 and X = 4.6650 [5]

iii)

For the third condition to be satisfied, it is necessary for the spread of the assets to exceed the spread of the liabilities.

This appears to be the case given that the liabilities occur in equal annual amounts at durations from 10 years to 20 years, whereas the assets are concentrated in two lumps at the two most extreme durations, 10 years and 20 years. [2]

[10 Marks]

Solution 6:

i) Bond yields are determined by investors' expectations of future short-term interest rates, so that returns from longer-term bonds reflect the returns from making an equivalent series of short-term investments. [2]

ii)

a) Let i_t be the spot yield over *t* years:

One year: yield is 8% therefore i_1= 0.08

Two years: $(1 + i_2)^2$= 1.08 x 1.07 therefore i_2= 0.074988
Three years: $(1 + i_3)^3$= 1.08x 1.07 x1.06 therefore i_3= 0.06997
Four years: $(1 + i_4)^4$= 1.08 x1.07 x1.06 x1.05 therefore i_4= 0.06494
(1 * 4 = 4)

[4]

b) Price of the bond is, 5 $[(1.08)^{-1} + (1.074988)^{-2} + (1.06997)^{-3}]$ + 105 $(1.06494)^{-4}$
= 13.03822 + 81.6373 = 94.67552 (1)

Find gross redemption yield from 94.67552 = $5a_4 + 100v^4$

At 7%; a_4= 3.3872; v^4= 0.76290gives RHS = 93.226
At 6%; a_4= 3.4651; v^4= 0.79209gives RHS = 96.5345

Interpolate between 6% and 7%. *i* = 0.06562 [4]

iii) Present value of the dividend is 4*v* calculated at 8% per annum effective is 3.70370.
Therefore forward price is
F = (400 - 3.70370) x 1.08 x1.07 = 457.96 [2]

[12 Marks]

Solution 7:

i) Working in lacs

Present value of the liability = 9 +12 x v x$\overline{a_1}$ at 9%
= 9 + 12 v $\frac{i}{\delta}$ v = 9+12 * 0.91743^2 * 1.044354
=19.54811

The assets upto (t+2) years from 1st January 2016 will have present value = $5v^2 a_t^{(2)}$

= $5\, v^2 \frac{i}{i^{(2)}} a_t$ = 5 x 0.84168 x 1.022015 xa_t = 4.301048 a_t

With *t*= 6, *PV* = 4.301048 x4.4859 = 19.2941

The next payment of 2.5 lacs at *t* = 6.5 is made at time 8.5 and has present value
= 2.5x $v^{8.5.}$ =1.2018

This would make PV of assets > PV of liabilities (19.5)

⇨ Discounted payback period = 8.5 years.

[6]

ii) The income of the development is received later than the costs are incurred. Hence an increase in the rate of interest will reduce the present value of the income more than the present value of the outgo. Hence the DPP will increase.

[2]

[8 Marks]

Solution 8:

The current price of stock:

P = 1,00,000 * 0.08 * $a^{(2)}_{15}$ + 1,00,000 * v^ 15 @ 7% (i^2 = 0.0688161)

= 8000*9.1079*1.017204+ 100000*0.36245

= INR 1,10,361

The forward can be calculated using the equation:

K = (P-I) exp(δT)

Where I is the PV, at the risk free rate, of the coupon payments due during the term of the forward contract and δ is the risk- free force of interest. Therefore,

K = (1,10,361 - 8,000* $a^{(2)}_{6}$) exp(6δ) @ δ=6% (equivalent i = 0.061836, i^2 = 0.06091)

K = (1,10,361 – 8,000 * 4.96352) * 1.4333 = 1,01,270

[5 Marks]

Solution 9:

i) Working in monetary values at t=0

$$25{,}000 = 10{,}000\left(v\,\frac{170.7}{183.3} + v^2\,\frac{170.7}{191.0} + v^3\,\frac{170.7}{200.9}\right)$$

Where v = $\frac{1}{(1+i')}$ where i' is the real rate of return

At 4% RHS = 24770.94

At 3% RHS = 25241.25

By interpolation i' = 3.5%

[3]

ii) 25,000 = 10,000 a_3 at i%

⇨ a_3= 2.5

⇨ $Hence$ i = 9.7%

[2]

[5 Marks]

Solution 10:

i) IRR :

Sheep rearing

The IRR i is the solution of the equation of value

$20{,}000 = 1{,}100 * a_{\overline{20}|} + 20000 * v^{20}$

@ 5%, RHS = 21,246.22
@ 6%, RHS = 18,853.00

By Interpolation, i = 5.5%

Goat breeding
The IRR i is the solution of the equation of value
$20{,}000 = 900 * a_{\overline{20}|} + 25000 * v^{20}$

@ 5%, RHS = 20,638.22
@ 5.5%, RHS = 19,323.57

By Interpolation, i = 5.24%

Forestry
The IRR i is the solution of the equation of value
$57{,}300 = 20{,}000 * (1+i)^{20}$

@ 5%, RHS = 53,065.95
@ 6%, RHS = 64,142.71

By Interpolation, i = 5.40% [6]

ii) Profitability of each of the project:

Sheep rearing
The 'surplus' income after paying the interest of the bank is INR 100 per annum,
So the profit is $100 * s_{\overline{20}|}$ @ 4%
= 100 * 29.7781
= 2,978.81

Hence the profit at the end of the project is 20,000 + 2,978.81 – 20,000 = INR 2,978.81

Goat breeding
A further INR 100 must be borrowed annually in arrear for 20 years,
And these loans grow at 5% p.a. compound

The total indebtedness at 20 years time is thus
$20{,}000 + 100 * s_{\overline{20}|}$ @ 5%

= 20,000 + 100 * 33.06595
= 23,306.59

Hence the profit at the end of the project is 25000 – 23,306.59
= INR 1,693.40

Forestry
After 20 years, the original debt of INR 20,000 will have grown to
= 20,000 * (1.05) ^ ^{20} = INR 53,065.95

The profit after 20 years is
= 57,300 – 53,066 = INR 4,234

Hence the highest profit is obtained from forestry [7]

[13 Marks]

Solution 11:
The repayments under each of the three loans can be found as follows:

Loan 1: P1 $a_{\overline{10}|}$ @ 10% = 6,00,000 , P1 = 6,00,000/6.1446 = INR 97,647.24

Loan 2: P2 $a_{\overline{5}|}$ @ 9.2025% = 250000 * v ^ $^{0.5}$, P2 = 2,50,000/(1.045* 3.8693) = INR 61,829

Loan 3: P3 $a_{\overline{5}|}$ @ 9.5% = 150,000 , P3 = 1,50,000/3.8397 = INR 39,065.55

The amount of capital outstanding under each of the three loans on 1 January, 2017 are:

Loan 1: P1 $a_{\overline{5}|}$ @ 10% = 97,647.24 * 3.7908 = INR 3,70,159.86

Loan 2: P2 $a_{\overline{2}|}$ @ 9.2025% = 61,829 * 1.7543 = INR 1,08,466.61

Loan 3: P3 *(1+i) ^ $^{0.5}$ $a_{\overline{5}|}$ @ 9.5% = 1.095^$^{0.5}$ * 1,50,000 = INR 1,56,963.37

The total capital outstanding is: 3,70,159.86+1,08,466.61+1,56,963.37 = INR 6,35,589.84

The annual repayments under the new loan can be found from the equation of value:

P * $a_{\overline{10}|}$ @ 8% = 6,35,589.84;
=> P = 6,35,589.84/6.7101 = INR 94,721.36

[9 Marks]

****