Institute of Actuaries of India

Subject CT8 – Financial Economics

March 2018 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) Framing reflects the way in which a choice is presented (or framed) to have a profound effect on the decision. It could be a wording on a question of gain or loss, words used in a question which can have an enormous impact.

For instance, say a question on exam paper can have significantly different answers if asked either of the following ways:

a. Was the paper lengthy?
b. Hope the paper wasn't too long? [2]

ii) Myopic loss aversion relates to the investors' aversion to the short term losses. It suggests that the investors are less risk-averse when faced with a multi period of gambles and that the frequency of choice or length of reporting period will also be influential.

For instance, an investor faced with a series of 10 half-yearly gambles are likely to be less risk averse compared to an investor facing a single gamble of 5 years. [2]

iii) Anchoring explains how people produce estimates on the basis of initial idea of an answer (referred to as 'anchor') and then adjust away from his initial anchor to arrive at their final judgment.

It suggests that the people base their perceptions on past experience or expert opinion which later they amend to allow for evident differences to current conditions.

[2]

[6 Marks]

Solution 2:

i) The strong form of EMH suggests that the market prices incorporate all information, both publically available and also that is available only to insiders.

If this exists, then even with insider information, the investors won't be able to generate higher returns. Hence, any such rules pertaining to company employees and management over ban in stock trading would be unnecessary in a strong form of market. [2]

ii) The semi strong form of market is said to be in existence if the market prices incorporate all publically available information.

The technical analysis relies on making trading rules based on historical price data to generate higher investment returns. Since in semi strong form of market, the prices already incorporate all public information, technical analysis won't help investors generating any additional returns. [2]

iii) In semi strong form of market, prices already incorporate all publically available information. However, the extent of public information might vary from investor to investor. For instance, different stock exchanges having different disclosure requirements are expected to have different levels of public information and hence, efficiency.

In addition, there could be additional costs involved in obtaining the public information accurately and quickly which otherwise would dilute the market efficiency. [2]

[6 Marks]

<u>Solution 3:</u>

i) Suppose X_t is a martingale with respect to a measure P, that is for any $t < s$,
$E_P\ [X_s | F_t] = X_t$ and the volatility of X_t is always non zero.

Suppose Y_t is another martingale with respect to P. Then, the martingale representation theorem states that there exists a unique previsible process ϕ such that
$Y_t = Y_0 + \int_0^t \phi_s\, dX_s$
Or, $dY_t = \phi_t\, dX_t$ [3]

ii) Given that $dS_t = \mu S_t\, dt + \sigma S_t\, dB_t$
By Ito's lemma, the SDE for this process is:

$d(\log S_t) = 1/S_t\, dS_t + ½\ (-1/S_t^2)(dS_t)^2$
$= 1/S_t(\mu S_t\, dt + \sigma S_t\, dB_t) – ½\ S_t^2(\mu S_t\, dt + \sigma S_t\, dB_t)^2$
$= (\mu\, dt + \sigma\, dB_t) – ½\ \sigma^2 d_t$
$= (\mu - ½\ \sigma^2)\, dt + \sigma\, dB_t$
Integrating this equation between n limits of s = 0 and s= t, we get
$[\log S_t] = (\mu - ½\ \sigma^2) \int_0^t ds + \sigma \int_0^t dBs$
It implies,
$\log S_t – \log S_0 = (\mu - ½\ \sigma^2)t + \sigma\, dB_t$
Hence, $S_t = S_0\, e^{(\mu - ½\ \sigma 2)t + \sigma Bt}$

[4]

[7 Marks]

Solution 4:

i) The first order stochastic dominance theorem states that assuming an investor prefers more to less, A will dominate B if:

$F_A(x) <= F_B(x)$, for all x, and
$F_A(x) < F_B(x)$, for some value of x

This means that the probability of portfolio B producing a return below a certain value never less than the probability of portfolio A producing a return below the same value, and exceeds it for at least some value of x.

The second order stochastic theorem applies when investor is risk-averse, as well as preferring more to less. In this case, the condition for A to dominate B is that

$\int_a^x F_A(y)\,dy <= \int_a^x F_B(y)\,dy$, for all x
$\int_a^x F_A(y)\,dy < \int_a^x F_B(y)\,dy$, for some value of x
It suggests that a risk-averse investor will accept a lower probability of a given extra return at a low absolute level of return in preference to the same probability of extra return at a higher absolute level. So the potential gain is valued less than a potential loss of the same amount.

[3]

ii)

a) Variance = $\int_{-\infty}^{\infty}(\mu - x)^2 \; f(x)\,dx$
In the given case, Rx ~ exp(λ)

Given, Mean = 4, i.e. $1/\lambda = 4$
Hence, $\lambda = 0.25$

Variance = $1/\lambda^2 = 16$ [2]

b) Downside Semi Variance = $\int_{-\infty}^{\mu}(\mu - x)^2 \; f(x)\,dx$

(b)

Downside Semi Variance $= \int_{-\infty}^{4} (4-x)^2 f(x)\, dx$

$= \int_0^4 (16 + x^2 - 8x)\, \lambda e^{-\lambda x}\, dx$

Now, $\int_b^a x e^{-\lambda x}\, dx = \frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \Big|_b^a$

$\therefore$,

$$DSV = \Big[-16 e^{-\lambda x} - 8\lambda\left(\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right) + \lambda\left(\frac{x^2 e^{-\lambda x}}{-\lambda} - \int \frac{2x e^{-\lambda x}}{-\lambda}\, ds\right)\Big]_0^4$$

$$= \Big[-16 e^{-\lambda x} + 8 x e^{-\lambda x} + \frac{8 e^{-\lambda x}}{\lambda} - x^2 e^{-\lambda x} + 2\left(\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right)\Big]_0^4$$

$= 16 - 32 e^{-1}$

(c) Shortfall probability

$\int_0^k \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda x}\right]_0^k$

$= 0.39347$

[3+2=5]

d)

(d) Expected Shortfall

$$\int_0^k x\lambda e^{-\lambda x}\,dx = \left[-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right]_0^k$$

$$= -ke^{-\lambda k} - \frac{e^{-\lambda k}}{\lambda} + \frac{e^{-0}}{\lambda}$$

Shortfall given default prob,

$$= k - \frac{\frac{1}{\lambda} - e^{-\lambda k}\left(k+\frac{1}{\lambda}\right)}{1-e^{-\lambda k}}$$

$$= 2 - \frac{4-6e^{-0.5}}{1-e^{-0.5}} = 1.083$$

[3]

[13 Marks]

Solution 5:

i) Let the proportion invested in Asset i be x*i*, with expected return Ei, Variance Vi and correlation as ρ12. Assume E to be the return on the portfolio of three assets and let λ and μ be the Lagrange multipliers. Then the Lagrangian function W satisfies:

$W = \Sigma i=1 \text{ to } 3\ \ x_i^2V_i + 2\ \rho_{12}\sigma_{12}x_1x_2 - \lambda(E_1x_1 + E_2x_2 + E_3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$

$= 36\ x_1^2 + 144\ x_2^2 + 324\ x_3^2 + 72x_1x_2 - \lambda\ (4x_1 + 6x_2 + 8x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$

[3]

ii)

(b)

$$\frac{\partial W}{\partial x_1} = 0 \Rightarrow 72x_1 + 72x_2 - 4\lambda - \mu = 0 \quad (1)$$

$$\frac{\partial W}{\partial x_2} = 0 \Rightarrow 288x_2 + 72x_1 - 6\lambda - \mu = 0 \quad (2)$$

$$\frac{\partial W}{\partial x_3} = 0 \Rightarrow 648x_3 - 8\lambda - \mu = 0 \quad (3)$$

$$\frac{\partial W}{\partial \lambda} = 0 \Rightarrow 4x_1 + 6x_2 + 8x_3 = 7 \quad (4)$$

$$\frac{\partial W}{\partial \mu} = 0 \Rightarrow x_1 + x_2 + x_3 = 1 \quad (5)$$

from (1), $\mu = 72x_1 + 72x_2 - 4\lambda$ (6)

Putting (6) into (2),

$$288x_2 + 72x_1 - 6\lambda - (72x_1 + 72x_2 - 4\lambda) = 0$$

$$\Rightarrow 216x_2 - 2\lambda = 0$$

or $\lambda = 108x_2$ (7)

Putting (6) & (7) into (3)

$$648x_3 - 8(108x_2) - (72x_1 + 72x_2 - 4(108x_2)) = 0$$

$$\Rightarrow 648x_3 - 504x_2 - 72x_1 = 0 \quad (8)$$

Also, $x_3 = 1 - x_1 - x_2$ (9)

Solving (9),

$$4x_1 + 6x_2 + 8(1 - x_1 - x_2) = 7$$

$$\Rightarrow x_2 = 0.5 - 2x_1 \quad (10)$$

Substituting (10) & (9) into (8),

$$648(1 - x_1 - (0.5 - 2x_1)) - 504(0.5 - 2x_1) - 72x_1 = 0$$

$$\Rightarrow x_1 = -0.04545$$

Hence, $x_2 = 0.59091$

$x_3 = 0.45454$

[5]

[8 Marks]

Solution 6:

i) The forward price is given by $F = S \cdot exp(rt)$ where S is the stock price, t is the delivery time and r is the continuously compounded risk-free rate of interest applicable up to time t.

Put-call parity states that: $c + K \cdot exp(-rt) = p + S$ where c and p are the prices of a European call and put option respectively with strike K and time to expiry t and S is the current stock price.

To compute F, we need to find S and r. t is given to be 0.25 years.

Substituting the values from the first two rows of the table in the put-call parity, we get two equations in two unknowns (S and r):

$$13.334 + 70 \cdot exp(-0.25r) = 0.120 + S$$

$$8.869 + 75 \cdot exp(-0.25r) = 0.568 + S$$

Solving the simultaneous equations for S and r, we get:

S = 82 and r = 7%

Therefore, we get the forward price $F = 82 \cdot exp(0.07 * 0.25) = 83.45$

[4]

ii) Let the (continuously compounded, annualized) rate of interest over the next k months be r_k. Then the required forward rate r_F can be found from:

$\exp(r_6*0.5) = \exp(r_3*0.25)*\exp(r_F*0.25)$ or $2*r_6 = r_3 + r_F$

We know that r_3 = 7%.

To find r_6, we substitute values from the last row in the put-call parity relationship and S = 82:

$2.569 + 90*\exp(-0.5*r_6) = 7.909 + 82$

Therefore, r_6 = 6% and r_F = 5% [2]

iii) Using the put-call parity for each row in the given table, we get:

$$6.899 + a*\exp(-0.07*0.25) = 1.055 + 82$$

$$b + 80*\exp(-0.07*0.25) = 1.789 + 82$$

$$2.594 + 85*\exp(-0.07*0.25) = c + 82$$

Solving individually, we get:

a = 77.5

b = 5.177

c = 4.119

[3]

[9 Marks]

Solution 7:

i) Since interest rates are assumed zero, the risk-neutral up-step probability is given as:

$q = (1-d) / (u-d)$

where u and d are the sizes of up-step and down-step respectively

For a recombining tree, d = 1/u.

Substituting d = 1/u in the expression for q and simplifying, we get:

q = (1-1/u) / (u – 1/u) = 1 / (u+1)

For no-arbitrage to hold, we must have u > 1 > d.

Then, u > 1 => u + 1 > 2 => q = 1 / (u+1) < ½. Hence proved.

[3]

ii) As derived for part (a), q = 1 / (u+1)

Substituting q = 1/3 and solving for u, we get: u = 2

For the calibration of a recombining binomial tree, we know that $u = exp(\sigma \cdot \sqrt{\Delta t})$

Using u = 2 and Δt = 1/12, we can solve for σ to get σ = 240.11% [2]

iii) Since each step is one month and the expiry of the derivative is one year from now. Therefore, a 12-step recombining binomial tree needs to be created, i.e. n = 12.

Further, at time T = 12 months, the stock price will be $S_0 u^k d^{n-k}$ with risk-neutral probability ${}^nC_k\, q^k (1-q)^{n-k}$ where q, the up-step probability is 1/3, u, the up-step size is 2, and d = 1/u = ½.

We know that the derivative has a payoff $\sqrt{\frac{S_T}{S_0}}$ at time T = 12 months.

Thus, the current price of that derivative is: $P = \sum_{k=0}^{n} \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k! \cdot (n-k)!} q^k (1-q)^{n-k}$

Therefore, $P = \sum_{k=0}^{n} \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k! \cdot (n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^{n} \sqrt{u^k d^{n-k}} \cdot \frac{n!}{k! \cdot (n-k)!} q^k (1-q)^{n-k}$

$$P = \sum_{k=0}^{n} u^{\frac{k}{2}} d^{\frac{n-k}{2}} \cdot \frac{n!}{k! \cdot (n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^{n} 2^{\frac{k}{2}} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k! \cdot (n-k)!} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

$$P = \sum_{k=0}^{n} 2^{k-\frac{n}{2}} \cdot \frac{n!}{k! \cdot (n-k)!} \frac{2^{n-k}}{3^n} = \sum_{k=0}^{n} 2^{\frac{n}{2}} \cdot \frac{n!}{k! \cdot (n-k)!} \frac{1}{3^n} = 2^{\frac{n}{2}} \frac{1}{3^n} \sum_{k=0}^{n} \cdot \frac{n!}{k! \cdot (n-k)!}$$

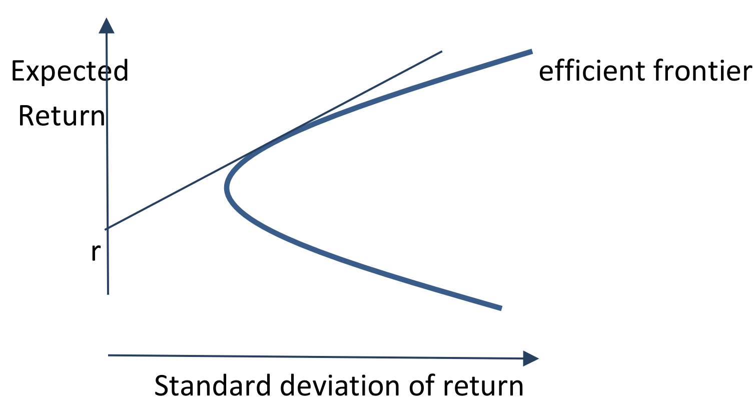
$$P = 2^{\frac{n}{2}} \frac{1}{3^n} 2^n = (\frac{2\sqrt{2}}{3})^n = (\frac{2\sqrt{2}}{3})^{12} = 0.49327$$

[5]

[10 Marks]

Solution 8:

i) The set of efficient portfolios in E – V space is known as the efficient frontier. A portfolio is efficient if the investor cannot find a better one in the sense that it has either a higher expected return and the same (or lower) variance or a lower variance and the same (or higher) expected return.



The portfolios which lie below the efficient frontier are sub-optimal as they do not provide enough return to compensate for the underlying risk.

[2]

ii) The security market line for any portfolio P is defined as

$E_P = r + (E_M - r)\ \beta_{P,}$

where

E_P is the expected return on Portfolio P

r is the risk free rate of return

E_M is the expected return on the market portfolio

β_P is the beta of the portfolio with respect to market portfolio

The capital market line for any portfolio P is defined as

$E_P - r = (E_M - r)\ \sigma_P / \sigma_M$

where

E_P is the expected return on Portfolio P on the efficient portfolio

r is the risk free rate of return

E_M is the expected return on the market portfolio

σ_P is the standard deviation of the return on Portfolio P

σ_M is the standard deviation of the return on market Portfolio

The capital market line relationship only holds for efficient portfolios which are a combination of the risk-free asset and the market portfolio whereas the security market line applies to any portfolios as well as individual securities. In other words, any security or a portfolio, whether efficient or not, would lie on the security market line. [2]

iii)

a) False [1]

b) False [1]

iv) As per CAPM, the market capitalization for Script A in the market portfolio would be 5%, i.e. in the proportion it is held by the investors.

[2]

[8 Marks]

Solution 9:

Both models are:

- continuous-time Markov models

- Ito processes
- one-factor models
- usually defined in terms of a standard Brownian motion under risk-neutral probability measure

The SDEs defining the two models are similar:

- Vasicek: $dr(t) = \alpha[\mu - r(t)]dt + \sigma dW(t)$
- Hull-White: $dr(t) = \alpha[\mu(t) - r(t)]dt + \sigma dW(t)$

Additionally, both models:

- imply the short-rate is mean-reverting
- imply the future short rate has a normal distribution
- allow negative values for the short rate
- are mathematically tractable, although Hull-White model is algebraically a bit more complicated

Key differences:

Vasicek model is time homogenous (μ constant), but Hull-White model is not (μ time-dependent).

Hull-White model has to be calibrated to match the current pattern of bond prices.

Hull-White model can provide a better fit to historical data.

[4 Marks]

Solution 10:

i) Two measures P and Q which apply to the same sigma algebra F are said to be equivalent if for any event E in F: P(E) >0 if and only if Q(E) > 0 where P(E) and Q(E) are the probabilities under P and Q respectively.

[1]

ii) Suppose that Z_t is a standard Brownian motion under P and that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\widetilde{Z_t} = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P, then there exists a previsible process γ_t such that $\widetilde{Z_t} = Z_t + \int_0^t \gamma_s ds$ is a Brownian motion under Q.

Note that the converse of the Cameron-Martin-Girsanov Theorem tells us that we can change the drift but not the volatility of the Brownian motion.

CMG Theorem is applied in the 5-step method is as follows:

- Step 1 of the 5-step method involves establishing the equivalent measure Q under which the discounted asset price process $D_t = e^{-rt}S_t$ is a martingale.
- This definition is equivalent to finding a measure with respect to which the expected share price evolves at the risk-free rate, i.e. it is a risk-neutral probability measure.
- CMG theorem assures us that such a measure exists and provides us with a way to change the measure by changing the drift.

[4]

[5 Marks]

Solution 11:

i) In the event of a default, the fraction of the defaulted amount that can be recovered through bankruptcy proceedings or some other form of settlement is known as the recovery rate.

[1]

ii) We can assume the government-issued bonds to be free of default risk. Therefore, the n-year discount factors DF_n can be implied as $DF_n = P_n/100$ where P_n is the price of the government-issued zero-coupon bond maturing n years from now.

$DF_5 = 0.778801$ and $DF_{10} = 0.576950$

We are given that the risk-neutral transition intensity for failure is of the form: $\lambda(t) = \alpha t$. The n-year risk-neutral probability of default p_n can therefore be derived as: $p_n = 1 - \exp(-\int_0^n \lambda(t)dt)$.

Substituting for λ(t) and integrating, we get: $p_n = 1 - \exp(-0.5\alpha n^2)$

Given the recovery rate δ, the risk-neutral expected payment at maturity will be $p_n\delta + (1 - p_n) \cdot 1$ which can be rewritten to $\delta + (1 - p_n) \cdot (1 - \delta)$. This should be multiplied with the discount factor above to give the current corporate bond price per unit notional, B_n.

$$[\delta + (1 - p_n) \cdot (1 - \delta)] \cdot DF_n = B_n$$

Substituting n = 5 and n = 10 gives us two equations in two unknowns (α, δ) as follows:

$$[\delta + \exp(-0.5\alpha \cdot 5^2) \cdot (1 - \delta)] \cdot 0.778801 = 0.771109$$

$$[\delta + \exp(-0.5\alpha \cdot 10^2) \cdot (1 - \delta)] \cdot 0.576950 = 0.554988$$

Solving these equations simultaneously, we get: α = 0.002 and δ = 60%.

[6]

iii) Using the value of α, p_7 can be found as $p_7 = 1 - \exp(-0.5 \cdot 0.002 \cdot 7^2) = 0.0478189$.

Similarly, DF_7 can be obtained from the given table as 0.657047

Then, using δ, p_7 and DF_7, B_7 can be computed as:

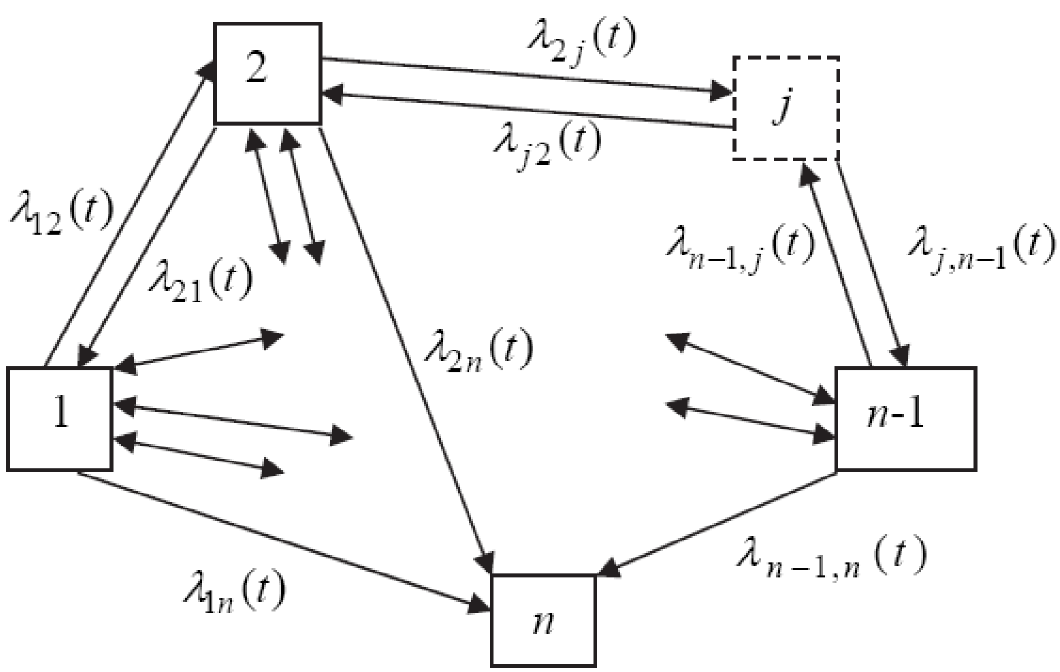
$$B_7 = [0.6 + \exp(-0.5 \cdot 0.002 \cdot 7^2) \cdot (1 - 0.6)] \cdot 0.657047 = 0.644479$$

[2]

iv) Compared to the simple default / no default two-state model, a more general and more realistic model has been developed by Jarrow, Lando and Turnbull, in which there are n states. The n states relate to n-1 possible credit ratings (when the company has not defaulted), and one default state.

(Transitions are possible between all states, except for the default state, which is absorbing (i.e. once the company has entered the default state, it cannot leave it.)

If the transition rate from state I to state j at time t is denoted by $\lambda_{ij}(t)$, where $\lambda_{ij}(t)$ is assumed to be deterministic, then the model can be represented by the following diagram:



[3]

[12 Marks]

Solution 12:

i) For a derivative whose price at time t is $f(t, S_t)$ where S_t is the price of the underlying asset,

- Delta is the rate of change of its price with respect to a change in S_t: $\Delta = \frac{\partial f}{\partial S_t}$
- Vega is the rate of change of its price with respect to a change in the assumed level of volatility of S_t: $\nu = \frac{\partial f}{\partial \sigma}$

[2]

ii) Put-call parity states that: $c + K^*\exp(-r\tau) = p + S$ where c and p are the prices of a European call and put option respectively with strike K and time to expiry τ and S is the current stock price.

Differentiating w.r.t. σ implies $\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma}$, i.e. the vegas are identical.

[1]

iii)

$$d_1 = \frac{log\frac{S}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

Therefore, $d_1 = 0.706241$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

Therefore, $d_2 = 0.456241$

$$c = S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)$$

Therefore, c = 9.652546

$$p = c + Ke^{-r\tau} - S$$

Therefore, p = 2.214017

[3]

iv) A portfolio for which the overall delta (i.e. weighted sum of the deltas of the individual assets) is equal to zero is described as delta-hedged or delta-neutral. Such a portfolio is immune to small changes in the price of the underlying asset.

A portfolio for which the overall vega (i.e. weighted sum of the vegas of the individual assets) is equal to zero is described as vega-hedged or vega-neutral. Such a portfolio is immune to small changes in the assumed level of volatility.

[2]

v) Let the required portfolio consist of x call options, y put options and z forwards.

The delta and vega for a forward are 1 and 0 respectively and there are no current cashflows.

Thus, for a single unit of each of them, we have:

	Present value / cashflow	Delta	Vega
Call option	c = 9.6525	Δ_c	V_c
Put option	p = 2.2140	Δ_p	V_p
Forward	-	1	-

Vega-neutrality: The vega of a forward is zero. For the portfolio must be vega-neutral, we must have: $x*V_c + y*V_p = 0$.
From part b, we have $V_c = V_p$. Therefore, $(x+y)*V_c = 0$. Therefore, $x+y = 0$. Therefore, $y = -x$.

Delta-neutrality:

We know that Δ of a forward is one. For the portfolio to be delta-neutral, we need: $x*\Delta_c + y*\Delta_p + z = 0$.

Also, $\Delta_p = \Delta_c - 1$ and $y = -x$. Therefore, on simplifying, we get: $x + z = 0$ or $z = -x$.

Overall portfolio:

Thus, we have $x = -y = -z$ and the total portfolio is to be worth $1000. So we must have:

$x*c + y*p + z*0 = 1000$. Therefore, $x*9.6525 - x*2.2140 = 1000$.

Therefore, $x = 134.4$, $y = z = -134.4$

So our portfolio must consist of:

- Long position of 134 call options
- Short position of 134 put options
- Short position of 134 forwards

[4]

[12 Marks]

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