Institute of Actuaries of India

Subject CT8 – Financial Economics

September 2016 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

1. Difficulty in testing EMH – There is a substantial body of literature proving the existence of mispricings, in contravention of EMH.
There is also a substantial body of literature proving various forms of EMH.

Testing Strong form - This is problematic, as it requires the researcher to have access to information that is not in the public domain. The studies of directors' share dealings suggest that, even with inside information, it is difficult to out-perform.

Testing Weak form - Studies have failed to identify a difference between the returns on stocks using technical analysis and those from purely random stock selection after allowing for transaction costs. No credible challenge has emerged to the EMH in its weak form.

Testing Semi Strong form. We will consider tests of this form of EMH in two categories, tests of informational efficiency and volatility tests:

Informational Asymmetry

Many studies show that the market over-reacts to certain events and underreacts to other events. The over/under-reaction is corrected over a long time period. If this is true then traders could take advantage of the slow correction of the market, and efficiency would not hold.

Examples

a. Over reaction
   i. past winners tend to be future losers and vice versa
   ii. stocks coming to the market by Initial Public Offerings and Seasoned Equity Offerings have poor subsequent long-term performance
   iii. Certain accounting ratios appear to have predictive powers, e.g. companies with high earnings to price, cashflow to price and book value to market value tend to have high future returns.

b. Under reaction
   i. Stock prices continue to respond to earnings announcements up to a year after their announcement.
   ii. Abnormal excess returns for both the parent and subsidiary firms following a de-merger.
   iii. Abnormal negative returns following mergers

c. Anomalies
   i. Anomalies attributed to behavioural patterns
   ii. Ability of accounting ratios to indicate out-performance, are arguably proxies for risk. Once these risks have been taken into account many

studies which claim to show evidence of inefficiency turn out to be compatible with the EMH.

Issues with Volatility Tests

Shiller found strong evidence that the observed level of volatility contradicted the EMH. There are criticisms to Siller's tests and these include:

- the choice of terminal value for the stock price
- the use of a constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of the series, ie the series may have stochastic trends that invalidate the measurements obtained for the variance of the stock price.

[9 Marks]

Solution 2:

i. P is used to denote the portfolio.

$$E_P = x_A \times E_A + x_B \times E_B$$

$$E_P = x_A \times E_A + (1 - x_A) \times E_B$$

$$x_A = \frac{E_P - E_B}{E_A - E_B} = \frac{0.08 - E_P}{0.02}$$

We have $\sigma_P^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2\rho_{AB}x_Ax_B\sigma_A\sigma_B$, substituting value of x_A & $\rho_{AB} = 0$,

$$\sigma_P^2 = x_A^2\sigma_A^2 + x_B^2\sigma_B^2 = \left(\frac{0.08 - E_P}{0.02}\right)^2 0.01 + \left(\frac{E_P - 0.06}{0.02}\right)^2 0.0144$$

$$\sigma_P = \sqrt{\left(\frac{0.08 - E_P}{0.02}\right)^2 0.01 + \left(\frac{E_P - 0.06}{0.02}\right)^2 0.0144}$$

$$\sigma_P = \sqrt{61{E_P}^2 - 8.32E_P + 0.2896}$$

Taking derivatives,

$$\frac{d\sigma_P}{dE} = \frac{1}{2\sqrt{61{E_P}^2 - 8.32E_P + 0.2896}}(122E_P - 8.32)$$

Equating to zero

$$E_P = \frac{8.32}{122} = 6.82\%$$

Efficient frontier is $\sigma_P = \sqrt{61{E_P}^2 - 8.32E_P + 0.2896}$ above mean return of 6.82%

[5]

ii. Investor prefers more to less and are risk averse **[1]**

iii. When $x_A = 1.5x_B$, $x_B = 0.4$ if portfolio has only A and B.

For this portfolio, $E_P = 6.8\%$

and $\sigma_P = \sqrt{61{E_P}^2 - 8.32E_P + 0.2896} = 7.68\%$

The new efficient frontier in presence of risk free asset will be a straight line passing through (r,0) and $(E,\sigma)_{x_A=1.5x_B}$

As the new efficient frontier should be tangent to the efficient frontier described in part (i) above, the slope of the new frontier should be $\left[\frac{d\sigma_P}{dE}\right]_{x_A=1.5x_B}$

i.e. $\frac{d\sigma_P}{dE} = \frac{1}{2\sqrt{61{E_P}^2-8.32E_P+0.2896}}(122E_P - 8.32) = \frac{-0.024}{2\times0.0768} = -0.1563$

We have

$\frac{(0.0768-0)}{(0.068-r)} = -0.1563$

Hence r = 55.93%

The equation of new frontier will be $\frac{\sigma}{E-0.5593} = -0.1563$

Hence $\sigma = 0.0874 - 0.1563E$ **[7]**

iv. Capital market line is in the form of $E - r = (E_m - r)\sigma/\sigma_m$

where 'm' denotes market portfolio with x_B=0.4

i.e. $E - 55.93\ \% = (6.8\% - 55.93\%)\sigma/7.68\%$

i.e. $E + 6.40\sigma = 55.93\ \%$

[2]

v. Beta can be calculated by solving $E_i - r = (E_m - r)\beta_i$

For Security A: $0.06 - 0.5593 = (0.068 - 0.5593)\beta_A$

Hence $\beta_A = 1.016$

For Security B: $0.08 - 0.5593 = (0.068 - 0.5593)\beta_B$

Hence $\beta_B = 0.976$ **[3]**

[18 Marks]

Solution 3:

i.

a. Suppose that initial values of X_1 and X_2 are a_1 and a_2. After a period of time T, X_1 has probability distribution N($a_1 + \mu_1$T, $\sigma_1{}^2$T) and X_2 has probability distribution N($a_2 + \mu_2$T, $\sigma_2{}^2$T)

Since X_1 and X_2 are independent normally distributed variables the distribution of X_1+ X_2 is given by N($a_1 + \mu_1 T + a_2 + \mu_2$T, $\sigma_1{}^2$T+$\sigma_2{}^2$T) = N(a_1 + a_{2+} (μ_1+ μ_2)T, ($\sigma_1{}^2$+$\sigma_2{}^2$)T)

This shows X_1+ X_2 follows generalized Weiner process with drift (μ_1+ μ_2) and variance rate $\sigma_1{}^2$+$\sigma_2{}^2$

[2]

b. Let the correlation coefficient be ρ. Then X_1+ X_2 follows N(a1 + a2 + (μ_1+ μ_2)T, ($\sigma_1{}^2$+$\sigma_2{}^2$ + 2 ρ$\sigma_1\sigma_2$)T

[2]

ii.

Define X_i, μ_i, σ_i as the stock price, expected return on stock price and volatility for stock i where i = X_1, X_2.

The SDE for the stocks will be (given that each follows Geometric Brownian Motion)

d$X_i = X_i\ \sigma_i\ d\ W_i\ + X_i\mu_i$dt where

Wi s are independent random samples from normal distribution

This implies dX_1 + dX_2 = $(X_1\mu_1 + X_2\mu_2)$dt + $X_1\ \sigma_1\ d\ W_1\ + X_2\ \sigma_2\ d\ W_2$

This shows that neither drift nor stochastic term correspond. Hence the value of the portfolio does not follow Geometric Brownian Motion.

[3]

iii. Suppose that the company's initial cash position be x.

The probability distribution after 1 year is given by N(x+ 4*0.5, 3*2)

i.e. X1 follows N(x+ 2, 6)

This implies [X1 – (x+ 2)]/6 follows N(0,1)

Now P(X1 <0) = P([X1 – (x+ 2)]/6 < -(x+2)/6) = 0.05 which implies Φ([-x-2]/6) = 0.05

This means [-x-2]/6 = -1.6449

Or x = 7.8694 **[4]**

iv. In this case

$$X_1 = X_0 e^{\mu t - ½\sigma^2 t + \sigma W t}$$

Where $W_t \sim N(0, \sigma^2 t)$.

P($X_1 < 1$) = P($LnX_1 < 0$)

Now LnX_1 follows N ($LnX_0 + \mu t - \frac{1}{2}\sigma^2 t$, $\sigma^2 t$)

Which means P(Z< [-($LnX_0 + \mu t - \frac{1}{2}\sigma^2 t$)]/ $\sigma\sqrt{t}$ = 0.05

This means Φ([-LnX_0- $\mu t + \frac{1}{2}\sigma^2 t$]/ $\sigma\sqrt{t}$ = 0.05

Or [-LnX_0- $\mu t + \frac{1}{2}\sigma^2 t$]/ $\sigma\sqrt{t}$] = -1. 6449

Or [-LnX_0- 2+18]/ 6 = -1.6449 or [-LnX_0+16]=-9.8694

Or [-LnX_0] = - 25.98694 or X_0 = exp(25.98694) **[4]**

[15 Marks]

Solution 4:

i. The process for forward bond yield is given by

$dy = \alpha y dt + \sigma y dz$ where z follows Weiner process i.e. N(0, $\sigma^2 t$)

The forward bond price is F(y)

Using Ito's Lemma,

$$dF(y) = \frac{\partial F}{\partial y}dy + \frac{1}{2}\frac{\partial^2 F}{\partial^2 y}(dy)^2 + \frac{\partial F}{\partial t}dt$$

$$= \frac{\partial F}{\partial y}(\alpha y dt + \sigma y dz) + \frac{1}{2}\frac{\partial^2 F}{\partial^2 y}(\alpha y dt + \sigma y dz)^2$$

$$= (\frac{\partial F}{\partial y}\alpha y + \frac{1}{2}\frac{\partial^2 F}{\partial^2 y}\sigma^2 y^2)dt + \frac{\partial F}{\partial y}\sigma y dz$$

[3]

ii. Assuming the process is a martingale, the drift term has to be zero which means

$$\frac{\partial F}{\partial y}\alpha y + \frac{1}{2}\frac{\partial^2 F}{\partial^2 y}\sigma^2 y^2 = 0$$

or $\frac{\partial F}{\partial y}\alpha y = -\frac{1}{2}\frac{\partial^2 F}{\partial^2 y}\sigma^2 y^2$

Or $\alpha = -\frac{1}{2}\frac{\partial^2 F}{\partial^2 y}\sigma^2 y^2 / \frac{\partial F}{\partial y} y = -\frac{1}{2}\frac{\partial^2 F}{\partial^2 y}\sigma^2 y / \frac{\partial F}{\partial y}$ **[3]**

iii. Yes the bond yield in general can be negative. This happens if the bond price at some time is higher than at par, in which case the bond yield will turn out to be negative. **[2]**

iv. The stock price cannot be negative, however, the return on the stock price can be negative as appropriately captured by the normal distribution. Let the continuously compounded rate of return per annum realized between time zero and T be λ.

$S_t = S_0 e^{\mu t - ½\sigma^2 t + \sigma W t}$. This implies rate of return λ follows N$((\mu - \frac{1}{2}\sigma^2), \sigma^2/T)$

[**2**]

v. Convexity is the relationship between the derivative price and the price of the underlying portfolio. Convexity of bond is a measure of the curvature or 2nd derivative of how the price of a bond varies with interest rate i.e. how the duration changes in response to interest rate changes. In this regard, modified duration is the first derivative of how the price of a bond varies in response to change in interest rate.

Let convexity be C, price of the bond be F, and interest rate be y.

Then C=(1/F) $\frac{\partial^2 F}{\partial^2 y}$,

modified duration D = -(1/F) $\frac{\partial F}{\partial y}$ or $\frac{\partial F}{\partial y} = -DF$.

Therefore CF = $\frac{\partial(-DF)}{\partial y}$ = (-D)(-DF)+ (-$\frac{\partial D}{\partial y}$)F

which implies C= D 2-($\frac{\partial D}{\partial y}$)

Hence $\alpha = \frac{1}{2}$FC σ^2 y/DF = $\frac{1}{2}$C σ^2 y/D = $\frac{1}{2}$[D ^{2}-($\frac{\partial D}{\partial y}$)] σ^2 y/D

[5]

[15 Marks]

Solution 5:

i. The Black Scholés differential equation is given by:

$\frac{\partial f}{\partial t}$ + rS $\frac{\partial f}{\partial S}$ + $1/2S^2\sigma^2 \frac{\partial^2 f}{\partial^2 S}$ = rf

The price of a forward contract at time t is given by

$f = S - Ke^{-r(T-t)}$

Where K is the delivery price of the forward contract and T is time to expiry.

This means $\frac{\partial f}{\partial t} = -rKe^{-r(T-t)}$ $\frac{\partial f}{\partial S}$ = 1, $\frac{\partial^2 f}{\partial^2 S} = 0$

Substituting in the above Black Schole's differential equation,

we have $-rKe^{-r(T-t)} + rS$ = r($-Ke^{-r(T-t)}$ +S) = rf

[3]

ii. $S_T = S_0 e^{\mu T - ½\sigma^2 T + \sigma WT}$

$\frac{1}{S_T} = \frac{1}{S_0 e^{(\mu - ½\sigma^2)T + \sigma WT}}$ = $S_0 e^{-(\mu - ½\sigma^2)T - \sigma WT}$

E(1/ST) = $[S_0 e^{-(\mu - ½\sigma^2)T} E(e^{-\sigma WT})]$

At time t, expected value of a derivative paying off $1/S_T$ at time T is given by

$[S_0 e^{-(\mu - ½\sigma^2)(T-t)}]\ e^{\frac{\sigma^2(T-t)}{2}}$

(If W_T is normally distributed with N(0, T), then e^{WT} is lognormally distributed)

$= [S_0 e^{-(r - ½\sigma^2)(T-t)}]\ e^{\frac{\sigma^2(T-t)}{2}}$

in a risk neutral world

The price of a security at time t is given by

$e^{-r(T-t)}$ E($1/S_T$) = $e^{-r(T-t)}[S_0 e^{-(r - ½\sigma^2)(T-t)}]\ e^{\frac{\sigma^2(T-t)}{2}}$

$= (1/S_t)\ e((^{-2r+\sigma^2)(T-t))})$ = f say

To be price of a derivative f should satisfy the expression $\frac{\partial f}{\partial t}$ + rS $\frac{\partial f}{\partial S}$ + $1/2S^2\sigma^2 \frac{\partial^2 f}{\partial^2 S}$

= rf

$\frac{\partial f}{\partial t} = (1/S_t)\ e^{(-2r+\sigma^2)(T-t))}$. (2r-$\sigma^2$)

$$\frac{\partial f}{\partial S} = -1/S^2 e^{(-2r+\sigma^2)(T-t))}$$

$$\frac{\partial^2 f}{\partial^2 S} = 2/S^3 e^{(-2r+\sigma^2)(T-t))}$$

Then $\frac{\partial f}{\partial t}$ + rS $\frac{\partial f}{\partial S}$ + $1/2S^2\sigma^2$ $\frac{\partial^2 f}{\partial^2 S}$ becomes

$(1/S_t)\ e^{(-2r+\sigma^2)(T-t))}.\ (2r\text{-}\sigma^2) + rS(-1/S^2 e^{(-2r+\sigma^2)(T-t))}) + 1/2S^2\sigma^2(2/S^3 e^{(-2r+\sigma^2)(T-t))})$

= f (2r-σ^2) – rf + fσ^2 = rf :- hence proved.

[7]

[10 Marks]

Solution 6:

i. Consider a stock whose current price is S_0 and an option whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level S_0u or move down to S_0d where u > 1 and d < 1.

Let the payoff be f_u if the stock price becomes S_0u and f_d if stock price becomes S_0d

Let us construct a portfolio which consists of a short position in the option and a long position in Δ shares. We calculate the value of Δ that makes the portfolio risk-free.

Now if there is an upward movement in the stock, the value of the portfolio becomes $\Delta S_0u - f_u$ and if there is a downward movement of stock, the value of the portfolio becomes $\Delta S_0d - f_d$

The two portfolios are equal if $\Delta S_0u - f_u = \Delta S_0d - f_d$

Or $\Delta = \frac{fu-fd}{S0u-S0d}$ so that the portfolio is risk-free and hence must earn the risk free rate of interest.

This means the present value of such a portfolio is $(\Delta S_0u - f_u)\exp(-rT)$

Where r is the risk free rate of interest.

The cost of the portfolio is $\Delta S_0 - f$

Since the portfolio grows at a risk free rate, it follows that

$(\Delta S_0u - f_u)\exp(-rT) = \Delta S_0 - f$

or $f = \Delta S_0 - (\Delta S_0u - f_u)\exp(-rT)$

Substituting Δ from the earlier equation simplifies to:

f = e^{-rT}[p f_{u} + (1-p) f_d] where p =[e^{rT} – d]/[u-d]

[5]

ii. The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in stock increases the value of call option and the value of put option decreases when the probability of stock price goes down

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret p as the probability of an up movement in the stock price. The variable 1-p is then the probability of a down movement such that the above equation can be interpreted as that the value of option today is the expected future value discounted at the risk free rate

[2]

iii.

The expected stock price $E(S_T)$ at time T = pS_0u + (1-p) S_0d 0.5

or $E(S_T)$ = p S_0(u-d) + S_0d ---0.5

Substituting p from above equation in (i) i.e. p = [e^{rT} – d]/[u-d]---1

We get $E(S_T)$ = e^{rT} S_0 ---0.5 -----1

i.e. stock price grows at a risk free rate or return on a stock is risk free rate

[3]

iv. In a risk neutral word individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk free interest rate. Hence value of an option is its expected payoff in a risk neutral discounted at risk free rate.

[2]

[12 Marks]

Solution 7:

1. (Equation of both the models)

2. Both are one factor model

3. BDT is not mean reverting whereas VM is.

4. Volatility parameter is constant for both the models.

5. Interest rates are strictly positive for BDT whereas they can be negative in VM.

6. Interest rates are log normally distributed for BDT whereas they are normally distributed for VM.

7. Both the models are simple to calibrate.

8. Both the models cannot be used to price complex derivatives

[4 Marks]

Solution 8:

i. There will be a Credit event if:

- Company fails to pay the coupon or capital
- Loss event
- Bankruptcy
- the credit rating of the company is changed.

Hence the rating has to stay at the current level i.e. AAA or BB or junk from the current time to the maturity date. The probability will then depend on the time to maturity (say T).

Probability of credit event if the bond is currently AAA is ($1 - 0.7^T$)

Probability of credit event if the bond is currently BB is ($1 - 0.6^T$)

Probability of credit event if the bond is currently junk is ($1 - 0.3^T$)

[2]

ii. To find the probability of default, we need the probability matrix for the third year i.e. we need P^3

$$P^3 = P.(P.P) = P\begin{pmatrix} 0.51 & 0.29 & 0.14 & 0.06 \\ 0.13 & 0.44 & 0.19 & 0.24 \\ 0.03 & 0.27 & 0.15 & 0.55 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.386 & 0.318 & 0.151 & 0.145 \\ 0.135 & 0.347 & 0.158 & 0.360 \\ 0.048 & 0.213 & 0.102 & 0.637 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Probability of default over three years given that the company is rated AAA now is 14.5%

The present value of the bond is given by $B = 100 \times (1 - 14.5\%) \times 1.07^{-3} = 69.79$

The yield on the ZCB $y = \left(\frac{100}{69.79}\right)^{\left(\frac{1}{3}\right)} - 1 = 1.1274 - 1 = 12.74\%$

Credit Spread = 12.74% - 7% = 5.74%

[6]

[8 Marks]

Solution 9:

Condition for First Order Stochastic Dominance A will be preferred to B

$F_A(x) \leq F_B(x)\ for\ all\ x\ and\ F_A(x) < F_B(x)\ for\ alteast\ one\ x$

Assumption: investor prefers more to less

Condition for Second Order Stochastic Dominance A will be preferred to B

$$\int_a^x F_A(y)dy \leq \int_a^x F_B(y)dy\ \ for\ all\ x\ and\ inequality\ holding\ strictly\ for\ alteast\ one\ x$$

Assumption: Investor prefers more to less and is risk averse

[4 Marks]

Solution 10:

The distribution of returns can be restated as 6% CERTAIN + 2.5% WITH 0.35 PROBABILITY

i.e. $100(212\ certain + 5\ @\ 35\%\ chance)$

i. mean value of the portfolio= $21200 + 0.35 \times 500 = 21375$ **[1]**

ii. Since the behaviour of contracts is independent of the other, the mean and variance are given by:

$Var(Portfolio) = 100 \times 5^2 \times 0.65 \times 0.35 = 568.75 = (23.85)^2$

$sd(portfolio) = 23.85$ **[1.5]**

iii. Since large number of contracts are involved, we can assume that the value of portfolio is normally distributed with the above calculated mean and variance.

Using normal approximation,

$VaR = 21375 + 1.645(23.85) = 21414.23$

This means that there is 95% chance that the value of portfolio is below INR 21414.23

[2.5]

[5 Marks]

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