Institute of Actuaries of India

Subject CT1 – Financial Mathematics

September 2016 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable

Solution 1:

i) Real rates should be used for calculations as the fees for college are expected to rise in future due to inflation. **[1]**

ii) Real rates should be used for evaluating cost of wrist watch while the money rates should be used for evaluating fixed INR 10,000 payments. **[1]**

iii) Real rates should be used as rent for care home will increase with inflation in future.

[1]

[3 Marks]

Solution 2:

i) $20{,}000 * (1 + i^4 / 4)^{4x17/12} = 26{,}000$

$(1 + i^4 / 4) = (26{,}000/20{,}000)^{12/68}$

$i^4 = (1.0474 – 1) * 4 = 18.96\%$ **[1]**

ii) $26{,}000 * (1 – d^2 / 2)^{2x17/12} = 20{,}000$

$(1 + d^2 / 2) = (20{,}000/26{,}000)^{12/34}$

$d^4 = (1 - .91156) * 2 = 17.69\%$ **[1]**

iii) $20{,}000 * (1 + i * 17/12) = 26{,}000$

$i = (26{,}000/20{,}000) * 12/17 – 1 = 21.18\%$ **[1]**

iv)

- If using simple interest, the rate of interest needs to be higher when compared to the compound interest in order to achieve the same overall amount.
- Interest rate will get lower with higher frequency of compounding.

[1]

[4 Marks]

Solution 3:

i)

- One party agrees to pay a floating rate and in return receives a fixed interest rate cash flow
- The other party agrees to pay a fixed interest rate and receive a floating interest rate cash flow.
- The fixed payments are at a constant rate for an agreed term.
- The floating payments will be linked to a short-term interest rate.

[2]

ii)

- The insurance company can purchase a long-dated fixed interest swap which involves paying floating cash flows and receiving fixed cash flows
- This would enable the insurance company to receive a series of fixed interest rate payments appropriate to immunize its liabilities.
- The commitment to pay floating would be met by the returns from the cash investment made by the insurance company. **[2]**

[4 Marks]

Solution 4:

i)

- The investor may need to reinvest the coupon payments. The terms that will be available for reinvestment are not known at the outset.
- For an investor who plans to sell before redemption, the ultimate sale price is not known at the outset.
- The real return (i.e. in excess of inflation) is uncertain. If inflation turns out to be higher than expected at the outset, the real returns from fixed interest bond will be lower than originally anticipated.
- Tax rates may change, affecting the income and capital proceeds received by the investor.

[2]

ii) 297,000 v^n = 11,000 $a_{7}^{(4)}$+ 220,000 v^7 at 6%

$$v^n = \frac{(11000 \times 1.022227 \times 5.5824) + (220000 \times 0.66509)}{297000}$$

1.06^{-n} = 0.703978

n = 6.024

[3]

[5 Marks]

Solution 5:

(1 + i) is log-normally distributed with mean 1.001 and variance 4×10^{-6}

⇨ 1.001 = exp $[\mu + \frac{\sigma^2}{2}]$

⇨ $4 \times 10^{-6} = \exp\,[2\mu + \sigma^2\,] \times \exp\,[\,\sigma^2 - 1]$

⇨ $\frac{4 \times 10^{-6}}{1.001^2} = \exp\,[\,\sigma^2 - 1]$

⇨ $\sigma^2 = 3.992 \times 10^{-6}$

⇨ μ= 0.0009975

⇨ $\log(1+i) \sim N(9.9975 \times 10^{-4}\ , 3.992 \times 10^{-6}\,)$

⇨ $\Pr\left(\ln(1+i) - 9.9975 \times 10^{-4}\ \middle|\sqrt{3.992 \times 10^{-6}}\,\right) \leq -1.645\Big)$ = 0.05

⇨ Pr (ln(1+i) $\leq -2.28921 \times 10^{-3}$) = 0.05

⇨ Pr(i≤ -0.002287) = 0.05

⇨ J = -0.2287%

[4 Marks]

Solution 6:

The relationship between i and i^p can be represented in a formula as follows:-

$(1 + i^p / p)^p = (1 + i)$

Also when p → ∞; the nominal rate of interest rate will become convertible continuously and is known as force of interest.

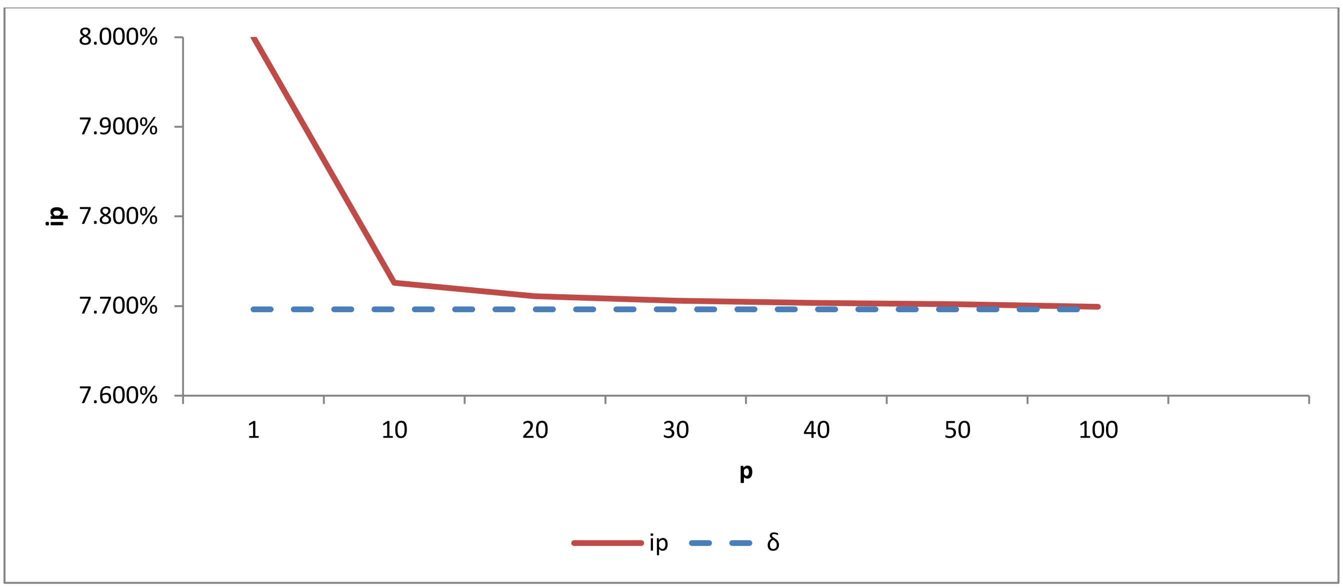
Calculating the force of interest:

$e^\delta = (1 + i) = 1.08$

$\delta = \ln(1.08) = 7.696\%$

The Force of interest will be the lowest value which will be attained by i^p

The graph depicting relationship between i^p and p will be:



[5 Marks]

Solution 7:

i) Present value of the liability

$= 1100\,v^5 + 1200v^{10} + 1300\,v^{15} + 1400v^{20}$ at 5%
$= 1100 \times 0.783526 + 1200 \times 0.613913 + 1300 \times 0.481017 + 1400 \times 0.376889$
$= 2{,}751.54$

Volatility
$= 5 \times 1100\,v^5 + 10 \times 1200v^{10} + 15 \times 1300\,v^{15} + 20 \times 1400\,v^{20}$

= 31,609.09
DMT = 31,609.09/2,751.54 = 11.48

Volatility = DMT/(1+i)
=11.48 /(1.05)
= 10.94

Convexity

= $1100 \times 5\times6\times v^{7} + 1200 \times 10\times11\times v^{12} + 1300 \times 15\times16\times v^{17} + 1400 \times 20\times21\times v^{22}$

=23452.48 + 73502.54 + 136124.60 + 201007.7
= 434,087.30

Convexity = 434,087.30/2,751.54
= 157.76

[6]

ii) The present value of the assets:4823 $\times v^{11.5}$ = 2,751.949

- The Present value of assets and liability is the same.
- The duration of the assets and the liability are the same.
- The convexity of the assets will be less than the convexity of the liability because the asset cash flow falls between the liability cash flow.

Redington's theory requires the opposite to be true for immunization. So this portfolio is reverse immunized, i.e. a small change in interest rates in either direction will lead to a deficit **[3]**

[9 Marks]

Solution 8:

i) The cash available for investment now at a constant force of interest of 5% p.a.

= 100,000 $\times v^{8}$ = INR 67,032

The price per unit nominal of the 20 year stock at $\delta = 0.05$

= v^{20} = 0.367879

:-Suppose the company buys a nominal amount of X of the 20-year stock at a cost

of 0.367,879 X

:-An amount (67032 - 0.367879 X) is held in cash

Cash has discounted mean term zero. The discounted mean term of company's assets is:

$= \frac{(0.367879 \times 20) + (67032 - 0.367879\,X) \times 0}{0.367879\,X + (67032 - 0.367879X)}$

$= \frac{7.357580\,X}{67032}$ = 8 (term of the liability)

- ⇨ X = 72,885
- ⇨ The company should buy INR. 72,885 nominal of 20 year zero coupon bond at the cost of 72885 × 0.367879 = INR 26,813
  &
  Hold INR 40,219 in cash **[6]**

ii) At a force of interest δ = 3% pa , the difference between company's assets and liabilities is
VA - VL
= 72885 ×$e^{-20\delta}$ + 40219 – 100000 ×$e^{-8\delta}$
= 1556 **[2]**

[8 Marks]

Solution 9:

i) Let X be the price paid by the investor A for the bond

X = 60 ×$a_{10}^{(2)}$ + 1000 ×v^{10} - 0.4×(1000 – X) v^{10} at 10%

X = INR 720.04

Let Y be the price paid by the investor B

Y = 60 × $a_{5}^{(2)}$ + 1000 v^{5} - 0.4×(1000 – Y) v^{5} at 10%
Y =INR 805.65

Investor A has made a capital gain of INR 805.65 – 720.04 = INR 85.61 on selling the bond

He pays capital gain tax of 40% ×85.61 = 34.24 on the date of sale.

Net proceeds = 805.65 -34.24 = 771.41

Yield => 720.04 = 60 × $a_{5}^{(2)}$ + 771.41 ×v^{5} at rate i

By interpolation the net annual yield of Investor A is arrived to be => I = 9.7%

[6]

ii) Let Net proceeds to investor A on the sale of the bond be Zin order for him to obtain a net yield of 10% p.a

720.04 = 60 × $a_{5}^{(2)}$ + Z ×v^{5} at 10%

Z = 784.39

The price paid by B is such that, P – 0.4 (P – 720.04) = 784.39

- ⇨ P = 827.29
- ⇨ Hence the net proceeds to investor B on redemption of the bond are
- ⇨ 1000 – 0.4 (1000 – 827.29) =930.92
- ⇨ Net yield Can be arrived by the following steps

$827.29 = 60 \times a_5^{(2)} + 930.92 \times v^5$ at rate i

By interpolation the net annual yield of Investor b is arrived to be = 9.50%

[5]

[11 Marks]

Solution 10:

i) $V(1) = 1.08^{-1} = 0.925926$

$V(2) = 1.08^{-1} \times 1.07^{-1} = 0.86535$

$V(3) = 1.08^{-1} x 1.07^{-1} x 1.06^{-1} = 0.81637$

$V(4) = 1.08^{-1} x 1.07^{-1} x 1.06^{-1} x 1.05^{-1} = 0.77749$

$P = 5 \times (0.925926 + 0.86535 + 0.81637 + 0.77749) + 100 \times 0.77749$

$= 94.6747$

Find *I* such that $94.6747 = 5\ a_4 + 100\ v^4$

i = 6.5% RHS = 94.8613

i = 7% RHS = 93.2256

By interpolation *i* = 6.55%

[7]

ii) The forward rates are falling with term. The gross redemption yield is a weighted average of those falling forward rates. It is therefore higher than the four year forward rate.

[2]

[9 Marks]

Solution 11:

The three options are evaluated as follows:

a) Landline phone

It is straight forward to evaluate this option as the scheme offers unlimited internet access.

Present value = $500 + 12 * 750 * a^{12}_{5@ 8\%}$

$a_{5@ 8\%} = 3.9927$

$a^{12}_{5@ 8\%} = a_{5@ 8\%} \backslash i^{12} = 3.9927 * (0.08 / .077208) = 4.1371$

Present value = 500 + 12 * 750 * 4.1371 = INR 37,734

b) Mobile Data

Evaluation of cost will involve fixed monthly rental and expected extra charge due to excess consumption.

Present value of monthly rental =12 * 600 * $a^{12}_{5@ 8\%}$ = 12 * 600 * 4.1371 = 29,787

We need to derive expected excess consumption.

Data consumption will follow pattern: $1{,}024\,(1 + 1.02 + 1.02^2 + \ldots + 1.02^{5*12-1})$

Month in which data consumption will cross 2GB limit can be determined with equation:

$1024 * 1.02^{m-1} = 2 * 1024 \Rightarrow m = 36.002$; hence the excess data cost will start from 37th month

Excess data is charged at INR 1 / MB. Equation to calculate excess data charges will be:

$= [(1024 * 1.02^{36} – 2048) + (1024 * 1.02^{37} – 2048) + \ldots + (1024 * 1.02^{59} – 2048)]$

$= [(1024 * 1.02^{36} + 1024 * 1.02^{37} + \ldots + 1024 * 1.02^{59}) – 2048 + 2048 + \ldots + 2048]$

Monthly effective rate = $1.08^{1/12} = 1.0064$

Evaluating its present value

$= [(1024 * 1.02^{36}/1.0064^{37} + 1024 * 1.02^{37}/1.0064^{38} + \ldots + 1024 * 1.02^{59}/1.0064^{60}) – 2048/1.0064^{37} + 2048/1.0064^{38} + \ldots + 2048/1.0064^{60}]$

$= 1024 * 1.02^{36}/1.0064^{37} * [(1.02/1.0064)^{24} – 1) / ((1.02/1.0064) - 1)] – 2048/1.0064^{37} * [(1 – 1.0064^{-24})/(1 - 1.0064^{-1})]$

= 46,400 – 36,107 = 10,293

Total present value of option = 29,787 + 10,293 = INR 40,080

c) Internet Dongle

Evaluation of cost will involve fixed installation charges, monthly rental and expected extra charge due to excess consumption.

Present value of monthly rental = 1,500 + 12 * 650 * $a^{12}_{5@ 8\%}$ = 1500 + 12 * 650 * 4.1371 = 33,769

Month in which data consumption will cross 3 GB limit can be determined with equation:

$1024 * 1.02^{m-1} = 3 * 1024 \Rightarrow m = 56.478$; hence the excess data cost will start from 57th month

Excess data is charged at INR. 1.5 / MB. Equation to calculate excess data charges will be:

$= [(1024 * 1.02^{56} – 3072) + (1024 * 1.02^{57} – 3072) + … + (1024 * 1.02^{59} – 3072)] * 1.5$

$= [(1024 * 1.02^{56} + 1024 * 1.02^{57} + … + 1024 * 1.02^{59}) – (3072 + 3072 + … + 3072)] * 1.5$

Evaluating its present value

$= 1.5 * 1024 * 1.02^{56}/1.0064^{57} * [(1.02/1.0064)^{4} – 1) / ((1.02/1.0064) - 1)] – 1.5 * 3072/1.0064^{37} * [(1 – 1.0064^{-4})/(1 - 1.0064^{-1})]$

= 13,211 – 12,691 = 520

Total present value of option = 33,769 + 520 = INR 34,289

Hence the cheapest option for Mr. Sahil would be to purchase Internet Dongle.

[13 Marks]

Solution 12:

The Loan schedule of Mr. Brown will need to consider different time periods based on the loan amount, interest rates and installment limits.

Monthly effective interest rate for 9.5% = 0.7592%

Monthly effective interest rate for 9% = 0.7207%

6^{th} Installment

Loan outstanding = 50% of total loan = 50% * 7,000,000 = 3,500,000

First year installment will be paid at interest only portion = 0.7592% * 3,500,000 = 26,570

51^{st} Installment

This installment will occur after possession of the home and hence the loan would be fully available to Mr. Brown and installments will include capital repayments as well. The installment amount will be 55,000.

We first determine the level installments due in third leg of the loan repayments.

Equation of the loan will be:

$7,000,000 = 55,000 * a^{12}_{5@9.5\%} * 12 + v^{5} * X * a^{12}_{2@9.5\%} * 12 + v^{7} * X * a^{12}_{10@9\%} * 12$

$a^{12}_{5@9.5\%} = 4.004$

$a^{12}_{2@9.5\%} = 1.822$

$a^{12}_{10@9\%} = 6.678$

X = INR 77,330

Loan outstanding before 51st installment (working with monthly annuity factors, interest rates):

$L_{51} = 55{,}000 * a_{46@.7592\%} + 77{,}330 * v^{46} * a_{24@.7592\%} + 77{,}330 * v^{70} * a_{120@.7207\%}$

$a_{46@.7592\%} = 38.704$

$a_{24@.7592\%} = 21.865$

$a_{120@.7207\%} = 80.139$

$L_{51} = 6{,}972{,}620$

Interest paid with 51st installment = 6,972,620 * 0.7592% = INR 52,933

<u>101st Installment</u>

This installment will occur after fixed payments of 55,000 are paid and now payments of 77,330 are being paid to bank.

Loan outstanding before 101st installment (working with monthly annuity factors, interest rates):

$L_{101} = 77{,}330 * a_{20@.7592\%} + 77{,}330 * v^{20} * a_{120@.7207\%}$

$a_{20@.7592\%} = 18.491$

$L_{101} = 6{,}757{,}140$

Interest paid with 101st Installment = 6,757,140 * 0.7592% = 51,297

<u>151st Installment</u>

Loan outstanding before 150th installment (working with monthly annuity factors, interest rates):

$L_{151} = 77{,}330 * a_{90@.7207\%}$

$a_{90@.7207\%} = 66.049$

$L_{151} = 5{,}107{,}593$

Interest paid with 151st Installment = 5,107,593 * 0.7207% = INR 36,812

Period	Loan Outstanding Beginning of year	Monthly installment	Interest due paid	Capital repaid	Loan Outstanding end of year
6th Installment	3,500,000	26,570	26,570	0	3,500,000
51st Installment	6,972,620	55,000	52,933	2,067	6,970,533
101st Installment	6,757,140	77,330	51,297	26,033	6,731,107
151st Installment	5,107,593	77,330	36,812	40,518	5,067,075

[13 Marks]

Solution 13:

i) Working in thousands ('000)

MWRR = $100 * (1+i)^5 + 50 * (1+i)^4 - 50 * (1+i)^3 + 100 * (1+i) = 246.7$

Try 6.5% LHS = 247.4345

Try 6.4% LHS = 246.621

MWRR = .064 + .001 * (246.7 – 246.621) / (247.4345 – 246.621) = 6.41% p.a.

Time	Total cash flow	Fund value just before investment	Fund value just after investment
1st Jan 2010	100,000	-	100,000
1st Jan 2011	50,000	101,000	151,000
1st Jan 2012	-50,000	157,300	107,300
1st Jan 2014	100,000	120,800	220,800
1st Jan 2015		246,700	246,700

TWRR = $(101/100) * (157.3/151) * (120.8/107.3) * (246.7/220.8) = (1+i)^5$

(1+i) = 1.0577 or I = 5.77% p.a.

[5]

ii) A significant investment had been made in year 2014 just before the outperformance of fund and withdrawn before the underperformance in equity in the year 2012; hence the MWRR return gives higher return than the TWRR.

[1]

iii) Working in thousands ('000)

Equity – calculating the cash flows

Time	Total cash flow in equity fund	Equity fund value just before investment	Value of fund assets just after investment
1st Jan 2010	60,000	-	60,000
1st Jan 2011	30,000	57,500	87,500
1st Jan 2012	-30,000	92,100	62,100
1st Jan 2014	60,000	67,200	127,200
1st Jan 2015	-	142,700	142,700

TWRR $_{\text{Equity}}$ => (57.5/60) * (92.1/87.5) * (67.2/62.1) * (142.7/127.2) = $(1+i)^5$

(1+i) = 1.0413 or i = 4.13% p.a.

Bonds – calculating the cash flows

Time	Total cash flow in bond fund	Bond fund value just before investment	Bond fund value just after investment
1st Jan 2010	40,000	-	40,000
1st Jan 2011	20,000	43,500	63,500
1st Jan 2012	-20,000	65,200	45,200
1st Jan 2014	40,000	53,600	93,600
1st Jan 2015	-	104,000	104,000

TWRR $_{\text{Bond}}$ => (43.5/40) * (65.2/63.5) * (53.6/45.2) * (104/93.6) = $(1+i)^5$

(1+i) = 1.0803 or i = 8.03% p.a.

Bond returns are higher than equity as they have steadily grown over a 5 year period, unlike equity which has seen volatility during the same period.

[6]

[12 Marks]

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