Actuarial Society of India

Examinations

November 2006

CT8 – FINANCIAL ECONOMICS

Indicative Solution

1.

- There are no transaction costs.
- Assets are infinitely divisible. For example, you can buy £1 worth of BAT stock.
- There is an absence of personal income tax. The major results of the model would hold if income tax and capital gains taxes are of equal size.
- An individual cannot affect the price of a stock by his buying or selling action.
- Investors are expected to make decisions solely on the basis of expected values and standard deviations of returns on their portfolios.
- Unlimited short sales are allowed.
- Unlimited borrowing and lending is available at the risk free rate.
- Investors are assumed to be concerned with the mean and variance of returns (or prices over a single period), and all investors are assumed to define the relevant period in exactly the same manner.
- All investors are assumed to have identical expectations with respect to the necessary inputs to the portfolio decision.
- All assets are marketable.
- Investors are risk averse.

[5]

2.

(i) (a) Specific Risk -- is the risk unique to a company or industry. The risk cannot be eliminated from the share, but it can be eliminated from a total portfolio by investing in a suitably diversified mix of shares of different types of companies.

(b) Beta -- is a measure of the volatility of a share price relative to the whole market. A share with a beta > 1 is expected to move more aggressively up or down than the market for a given market move. Conversely, a share with a beta < 1 will be expected to move more defensively. For a portfolio, its beta value is the weighted average of its constituent shares, weighted by market value of holdings.

(ii) This is the risk of the individual share relative to the overall market and it cannot be eliminated by diversification.

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3.

(i) ***Multifactor model***

The multifactor model attempts to explain returns on assets by relating them to a series of n factors known as indices:

$$R_i = a_i + b_{i,1} I_1 + \dots + b_{i,n} I_n + c_i \qquad \text{where:}$$

a_i, c_i are the constant and random parts of the return, specific to asset i

$I_1, I_2, \dots, I_n$ are the n indices explaining the returns on all the stocks

$b_{i,k}$ is the sensitivity of the return on stock i to factor/index k

$E[c_i]=0$

$cov[c_i,c_j]=0$ for all $i \neq j$

$cov[c_i,I_k]=0$ for all stocks and indices.

(ii) ***Three types of factor***

1. ***Macroeconomic*** - the factors would include some macroeconomic variables such as interest rates, inflation, economic growth and exchange rates.
2. ***Fundamental*** - the factors will be company specifics such as P/E ratios, liquidity ratios and gearing levels.
3. ***Statistical*** - the factors do not necessarily have a meaningful interpretation. This is because they are derived from historical data, using techniques such as principal components analysis to identify the most appropriate factors.

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4.

(i) Describe briefly what is meant by a short-rate model

Short-rate models are one of several approaches that can be used to model interest rates. This approach focuses on the "short rate" $r(t)$, which is the force of interest applicable at the current time t for "overnight" investments.

$r(t)$ is assumed to behave as an Itó process over calendar time t.

How are they used for pricing?

To use the model for pricing, we need to find the Itó process for the short rate under the risk-neutral probability measure Q.

The price at time t of a zero-coupon bond maturing at time T can then be found using the formula:

$$B(t,T) = E_Q\left[\exp\left(-\int_t^T r(u)du\right) \Big/ r(t)\right]$$

The price at time t of an interest-rate derivative with a payoff X at time T can be found using the formula:

$$V_t = E_Q\left[\exp\left(-\int_t^T r(u)du\right)X \Big/ r(t)\right]$$

(ii) Explain what is meant by a one-factor model

A one-factor model is one in which interest rates are assumed to be influenced by a single source of randomness.

The prices of all bonds (of all maturities) and interest-rate derivatives must therefore move together.

The randomness is usually modelled as an Itó process.

The stochastic differential equation for $r(t)$ has the following form under the real-world probability measure P :

$$dr(t) = a\,(t\,,\,r(t))\;dt + b\,(t\,,\,r(t))\;dW(t)$$

Where a ( ) and b ( ) are appropriately-chosen functions.

[7]

5.

(i) Put-call parity expresses a relationship between the price of a put option and the price of a call option on a stock where the options have the same exercise dates and strike prices.

(ii) Consider a portfolio A which contains one European call and an amount of cash $D + X\,e^{r(T-t)}$

where X = strike price

r = risk-free rate

$T - t$ = time to exercise of the option

D = present value of dividends payable

At the exercise date if the share price $S_T \geq X$ then call will be exercised and portfolio A will have a value of

$$D\,e^{r(T-t)} + S_T$$

If at T we have $S_T < X$ then the call will not be exercised and portfolio A will be worth

$$D\,e^{r(T-t)} + X$$

Now consider portfolio B consisting of one European put and a share.

At the exercise date if $S_T \geq X$ then the put will not be exercised and portfolio B will have value of

$$S_T + De^{r(T-t)}$$

If at the exercise date T, we have $ST < X$ then the put will be exercised and portfolio B will have a value of

$$X + De^{r(T-t)}$$

Clearly portfolios A and B have the same value in all circumstances at the exercise date T. Hence they must be equivalent at all earlier times $\Rightarrow$ the portfolios are of equal value

$$\therefore c + D + X\,e^{-r(T-t)} = p + S_t$$

c = value of European call with strike X and exercise date T

p = value of European put with strike X and exercise date T

S_t = value of stock at time t

(iii) Let D be the present value of dividends payable and consider

$$c + D + X\,e^{-r(T-t)} < p + S_t$$

then for some amount A

$$A + c + D + X\,e^{-r(T-t)} = p + S_t$$

Hence we can short one share and sell a put and receive $p + S_t$. At the exercise date we know the value of this portfolio is

$$\max[S_t + D\,e^{r(T-t)}, X + D\,e^{r(T-t)}].$$

However we know that the value of a portfolio invested in a European call and $D + X e^{r(T-t)}$ at time t will be worth

$\max[S_t + D e^{r(T-t)}, X + D e^{r(T-t)}]$.at T.

This is the same as the amount we must repay at time T.

Hence we are left with a profit of $A e^{r(T-t)}$

$\therefore$ strategy is

Short 1 share and sell a put.

Buy 1 call and put on deposit $A + D + X e^{r(T-t)}$

If the inequality is reversed also reverse investment (i.e. swap long positions for short positions and vice versa).

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6.

(i) ***Replicating portfolio of European call option***

We have two possibilities for the price at time 1:

$$S_1 = \begin{cases} S_0u & \text{if the price goes up} \\ S_0d & \text{if the price goes down} \end{cases}$$

We can hold an amount ϕ of the stock, and amount ψ of cash with the intention of replicating a derivative whose payoff is c_u if the stock price goes up and c_d if the stock price goes down.

At time 1 the portfolio has the value:

$$S_1 = \begin{cases} \phi S_0u + \psi e^r & \text{if the stock price went up} \\ \phi S_0d + \psi e^r & \text{if the stock price went down} \end{cases}$$

We can now solve the simultaneous equations:

$$\phi S_0u + \psi e^r = C_u$$

$$\phi S_0d + \psi e^r = C_d$$

Equation (1) gives $\phi = \dfrac{c_u - \psi e^r}{S_0 u}$. Substituting into equation (2) gives:

$$\left(\frac{c_u - \psi e^r}{S_0 u}\right) S_0 d + \psi e^r = c_d$$

giving $\psi = e^{-r}\left(\dfrac{c_d u - c_u d}{u - d}\right)$ and $\phi = \dfrac{c_u - \psi e^r}{S_0 u} = \dfrac{c_u - \dfrac{c_d u - c_u d}{u - d}}{S_0 u} = \dfrac{c_u - c_d}{S_0(u - d)}$

By the no-arbitrage principle, the value of this portfolio at time 0, V_0, must also be the value of the derivative contract at that time.

Finally, we are actually asked to replicate a European call with strike price of k. This implies that $c_u = uS_0 - k$ and $c_d = 0$ since we are told that $d\,S_0 < k < u\,S_0$. Substituting in our expressions for ϕ and ψ gives:

$$\phi = \frac{S_0 u - k}{S_0(u-d)}$$

and $$\psi = \left(\frac{-(S_0 u - k)d}{u-d}\right) = e^{-r}\left(\frac{S_0 u - k}{u-d}\right)d$$

(ii) ***Risk-neutral measure Q***

We want to show that: ϕ and ψ

$$V_0 = \phi S_0 + \psi = e^{-r} E_Q(C/F_0) = e^{-r}(q\,(S_0 u - k))$$

where C is the call option payoff at time 1 and q is the required (risk-neutral) probability of an upward stock price movement. This gives:

$$\phi S_0 + \psi = \left(\frac{S_0 u - k}{S_0(u-d)}\right)S_o - e^{-r}\left(\frac{(S_0 u - k)d}{u-d}\right) = e^{-r}(S_0 u - k)\left(\frac{e^r - d}{u-d}\right)$$

Comparing the final expression to $e^{-r} E_Q(C) = e^{-r}(q\,(S_0 u - k))$ gives us $q = \dfrac{e^r - d}{u-d}$

(iii) ***Risk-neutral versus real-world probabilities***

The probability measure Q was constructed in part (ii) so that the value of the derivative at time 0 was the discounted value, at the risk-free rate, of its expected payoff at time 1. Since the replicating portfolio has the same value as the option at both times, the portfolio must earn the risk-free rate of interest. The portfolio comprises of cash, which earns the risk-free, and stock.

In order that the portfolio earns the risk-free rate under Q, the stock itself must earn the risk-free rate, as we can verify:

$$E_Q[S_1/F_0] = S_0\,(qu + (1-q)\,d)$$

$$= S_0\left(\left(\frac{e^r - d}{u-d}\right) + \left(1 - \frac{e^r - d}{u-d}\right)d\right)$$

$$= S_0\left(\frac{e^r u - ud + ud - d^2 - e^r d + d^2}{u-d}\right)$$

$$= e^r S_0$$

Under the probability measure Q investors are therefore assumed to be risk-neutral, *ie* they demand no extra return from the stock even though it has higher risk (variance) than the cash.

The relationship of Q to the real-world probability measure P will depend on the preferences of investors. If, as is generally considered to be the case, investors are risk-averse, then the actual real-world probability p must be greater than the risk-neutral probability q, so that $E_p(S_1) > e^r S_0$. This must be so because the actual expected return must be higher than the risk-free rate to compensate them for the risk.

If the investors are actually risk-neutral in the real world, then $p = q$. While if they are risk-seeking, then $p<q$.

[13]

7.

(i) ***Differential equation***

$$dBt = r\, Bt\, dt$$

(ii) ***What is a self-financing portfolio?***

The changes in the value of a self-financing portfolio are due purely to the changes in the prices of the constituent assets, and not due to injections or withdrawals of money into or out of the portfolio.

If a portfolio of shares and cash has value f , *ie* $f = \phi_t S_t + \psi_t B_t$ then it will be self-financing if and only if:

$$df = \phi_t\, dS_t + +\psi_t\, dB_t$$

(iii) What is a previsible process?

A process is previsible if its value at time t can be deduced from the information that is known up to but not including time t.

(iv) ***Deduce the results***

Starting from $\phi_t S_t + \psi_t B_t = f(t, S_t)$we have:

$$d(\phi_t S_t + \psi_t B_t) = df(t, St)$$

Assuming the portfolio is self-financing, the left-hand side must be $\phi_t dS_t + \psi_t dBt$. So, applying Ito's lemma to the RHS, we get:

$$\phi_t dS_t + \psi_t dB_t = \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta S_t} dS_t + \frac{1}{2} \frac{\delta^2 f}{\delta S_t^2} (dS_t)^2$$

Now use the SDEs for the share price and the bond, ie $dS_t = S_t\,(\mu\, dt + \sigma\, dZ_t)$ and $dB_t = rB_t dt$. The former also implies that $(dSt)^2 = \sigma^2\, St^2 dt$ using the multiplication table for increments.

Therefore:

$$\phi_t S_t (\mu dt + \sigma dZ_t) dt + \psi_t r B_t dt = \frac{\delta f}{\delta t} dt + \frac{\delta f}{\delta S_t} S_t (\mu dt + \sigma dZ_t)_t + \frac{1}{2} \sigma^2 S_t^2 \frac{\delta^2 f}{\delta S_t^2} dt$$

i.e. $(\phi_t S_t \mu + \psi_t r B_t)dt + \phi_t S_t \sigma dZ_t = \left(\frac{\delta f}{\delta t} + \frac{\delta f}{\delta S_t} S_t \mu + \frac{1}{2}\sigma^2 S_t^2 \frac{\delta^2 f}{\delta S_t^2} \right) dt + \frac{\delta f}{\delta S_t} S_t \sigma dZ_t$

Comparing the *dZt* terms we must have:

$$\phi_t S_t \sigma = \frac{\delta f}{\delta S_t} S_t \sigma$$

and therefore:

$$\phi_t = \frac{\delta f}{\delta S_t}$$

Similarly, if we look at the *dt* terms we have:

$$\phi_t S_t \mu + \psi_t r B_t = \frac{\delta f}{\delta t} + \frac{\delta f}{\delta S_t} S_t \mu + \frac{1}{2}\sigma^2 S_t^2 \frac{\delta^2 f}{\delta S_t^2}$$

We have already established that $\phi_t = \frac{\delta f}{\delta S_t}$. So the terms containing μ cancel, giving:

$$\psi_t r B_t = \frac{\delta f}{\delta t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\delta^2 f}{\delta S_t^2}$$

By assumption, we know that $\phi_t S_t + \psi_t B_t = f$ and hence $\psi_t B_t = f - \frac{\delta f}{\delta S_t} S_t$

Substituting this into the left-hand side of the previous equation gives:

$$rf - rS_t \frac{\delta f}{\delta S_t} = \frac{\delta f}{\delta t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\delta^2 f}{\delta S_t^2}$$

which is equivalent to the equation given in the question.

[13]

8.

(i) Cov (A, B) = $\sigma_{A,B} = E[(A - E(A))(B - E(B))]$

$E(A) = 0.2 \times .18 + 0.3 \times .11 = 6.9\%$

$E(B) = 0.2 \times .13 + 0.3 \times .06 = 4.4\%$

$\sigma_{A,B}$ = 0.2(.031 ×(.064) + (0.011 ×0.106) + (0.3(.181 ×(-0.044) + (-.209) ×0.016)

= -0.001636 -.003392

= -0.003556

$\sigma_A^2 = E[(A - E(A))^2] = E[A^2] - E[A]^2$

$= 0.2(.1^2 + .08^2) + .3(.25^2 + .14^2) - 0.069^2 = .00328 + .02463 - .004761$

= .023149

$\sigma_B^2 = 0.2(.02^2 + .15^2) + 0.3(0^2 + .06^2) - .044^2$

= .00458 + .00108 -.001936

= .003724

Corr(A, B) = $\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} = \frac{-.003556}{.152148 \times .061025}$

$\tilde{=}$ .383

(ii) Assume proportion ." α". of assets are in asset A.

Let Portfolio be $P = \alpha A + (1 - \alpha) B$

Return on Portfolio is Rp

$V(Rp) = \alpha^2 \sigma_A^2 + (1-\alpha)^2 \sigma_B^2 + 2\alpha(1-\alpha) \sigma_{AB}$

$\frac{dV(R_p)}{d\alpha} = 2\alpha \sigma_A^2 + (1-\alpha)(-1) \sigma_B^2 + (1-2\alpha) \sigma_{AB}$

set = 0

$\Rightarrow 0 = \alpha(\sigma_A^2 + \sigma_B^2 - 2 \sigma_{AB})$

$$\alpha = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \cong 0.2142$$

$\therefore$ invest 21.42% of portfolio in asset A to get minimum risk portfolio.

(iii) No diversification benefits remain when the variance of the portfolio equals the variance from holding only asset B.

$\therefore$ when $V(R_p) = V(B)$

$\therefore$ i.e. $\alpha^2 \sigma_A^2 + (1-\alpha)^2 \sigma_B^2 + 2\alpha(1-\alpha) \sigma_{AB} = 0.003724$

$\Rightarrow \sigma_{AB} = 0.000486$

$$\Rightarrow \rho_{AB} = \frac{0.000486}{0.152148 \times 0.061025}$$

$= 0.052$

[13]

9.

(i) ***Solving the stochastic differential equation***

The question tells us to consider the function log *St* .

By Itô's lemma, the stochastic differential equation for this process is:

$$d(\log S_t) = \frac{1}{S_t}dS_t + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(dS_t)^2$$

$$= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) - \frac{1}{2S_t^2}(\mu S_t dt + \sigma S_t dB_t)^2$$

$$= (\mu dt + \sigma dB_t) - \frac{1}{2}\sigma^2 dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t$$

Integrating this equation between n limits of $s = 0$ and $s = t$, we get:

$$[\log S_t]_{s=0}^{s=t} = \left(\mu - \frac{1}{2}\sigma^2\right)\int_0^t ds + \sigma\int_0^t dB_s$$

$$\Rightarrow \log S_t - \log S_0 = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t$$

$$\Rightarrow S_t = S_0 e^{(\mu - 1/2\sigma^2)t + \sigma B_t}$$

(ii) ***Probability that the share price will exceed 110 (at the end of the period)***

We need to calculate:

$P(S_{6/12} > 110 \mid S_0 = 100)$

$$= P\left(\frac{S_{1/2}}{S_0} > \frac{11}{10}\right)$$

$$= P\left(e^{\sigma B_{1/2} + \frac{1}{2}(\mu - 1/2\sigma^2)} > \frac{11}{10}\right)$$

$$= P\left(\sigma B_{1/2} + \frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right) > \log\frac{11}{10}\right)$$

$$= P\left(0.1B_{1/2} + \frac{1}{2}\left(0.2 - \frac{1}{2}\times 0.1^2\right) > \log\frac{11}{10}\right)$$

$$= P(B_{1/2} > -0.022)$$

Since $B_{1/2} \sim N(0,0.5)$, this is

$$1-\Phi\left(\frac{-0.022-0}{\sqrt{1/2}}\right)=\Phi(0.031)=0.512 \text{ or } 51.2\%$$

[9]

10.

(i) ***The Vasicek model and its statistical properties***

This is a model used for modelling the short-rate of interest st $r(t)$.

It assumes that $r(t)$ has the dynamics of an Itó process (in fact, an Ornstein-Uhlenbeck process) under the risk-neutral probability measure Q.

The Vasicek model assumes the model $dr(t) = \alpha\,[\mu - r(t)]dt + \sigma\, dW(t)$, where $W(t)$ is standard Brownian motion.

The movements in the interest rate are therefore normally distributed and the parameter σ controls the volatility.

The parameter α is chosen to be in the range (0,1) , so that $r(t)$ is mean-reverting to the value constant value μ.

(ii)(a) ***Derive an equation for dUt***

Here finding the stochastic increment requires the product rule. (It is not obvious that this is allowed in stochastic calculus, and indeed it isn't in general. However, it is legitimate to use it in this situation, where one of the factors in the product is deterministic):

$$dUt = d(e^{\alpha t}r_t) = e^{\alpha t}dr_t + r_t\alpha e^{\alpha t}dt$$

Now we can substitute in for dr_t and simplify:

$$dUt = e^{\alpha t}\,(\alpha\,[b - r_t]dt + \sigma dBt) + r_t\alpha e^{\alpha t}dt$$

$$= \alpha e^{\alpha t}\, b\, dt + e^{\alpha t}\,\sigma dBt$$

(ii)(b) Solve the equation

We integrate both sides from 0 to t :

$$U_t - U_0 = \int_0^t \alpha b e^{\alpha s}ds + \int_0^t \sigma e^{\alpha s}dB_s$$

$$\Rightarrow U_t = U_0 + b\left(e^{\alpha t}-1\right) + \int_0^t \sigma e^{\alpha s}dB_s$$

(ii)(c) ***Show that***

Expressing the previous expression for U_t in terms of r_t , we have:

$$e^{\alpha t} r_t = r_0 + b\left(e^{\alpha t} - 1\right) + \int_0^t \sigma e^{\alpha s} dB_s$$

$$\Rightarrow r_t = b + e^{\alpha t}\left(r_0 - b\right) + \int_0^t \sigma e^{\alpha(s-t)} dB_s$$

(iii) ***Probability distribution of rt***

dBs ~ *N* (0, *ds*)

$$\sigma e^{\alpha(s-t)} dB_s \sim \mathrm{N}\left(0, \sigma^2 e^{2\alpha(s-t)} ds\right)$$

$$\int_0^t \sigma e^{\alpha(s-t)} dB_s \sim \mathrm{N}\left(0, \int_0^t \sigma^2 e^{2\alpha(s-t)} ds\right)$$

The distribution of r_t is given by:

$$r_t \sim \mathrm{N}\left(\mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right), \int_0^t \sigma^2 e^{2\alpha(s-t)} ds\right) = N\left(\mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right), \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right)$$

As t →∞ we get

$$r_t \sim \mathrm{N}\left(\mathrm{b}, \frac{\sigma^2}{2\alpha}\right)$$

(iii) ***Derive the conditional expectation***

We have:

$$E\left[r_t / F_s\right] \sim \mathrm{E}\left(\mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right) + \int_0^t \sigma e^{\alpha(u-t)} dB_u\right)$$

$$= \mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right) + \mathrm{E}\left(\int_0^t \sigma e^{\alpha(u-t)} dB_u / F_s\right)$$

$$= \mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right) + \mathrm{E}\left(\int_0^s \sigma e^{\alpha(u-t)} dB_u / F_s\right) + \mathrm{E}\left(\int_s^t \sigma e^{\alpha(u-t)} dB_u / F_s\right)$$

$$= \mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right) + \int_0^s \sigma e^{\alpha(u-t)} dB_u$$

We can now relate this to *rs* which we know from (i)(c):

$$= \mathrm{b} + \mathrm{e}^{-\alpha t}\left(r_0 - b\right) + e^{-\alpha(t-s)} \int_0^s \sigma e^{\alpha(u-t)} dB_u$$

using the result from part (i)(c) with *s* replaced by *u* and *t* replaced by *s*.

$$= e^{\alpha(s-t)}\left[r_s - b - (r_0 - b)e^{-\alpha s}\right] + b + e^{-\alpha s}(r_0 - b)$$

$$= e^{\alpha(s-t)}r_s + b\left(1 - e^{\alpha(s-t)}\right)$$

[18]

[Total 100]