Actuarial Society of India

Examinations

November 2006

CT1 –Financial Mathematics

Indicative Solution

Q.1

I. An insurance company earned a simple rate of interest of 8% over the last calendar year based on the following information:

Item	Rs.
Assets, beginning of year	25,000,000
Sales revenue	X
Net investment income	2,000,000
Salaries paid	2,200,000
Other expenses paid	750,000

All cash flows occur at the middle of the year. Calculate the effective yield rate.

Solution:

$$2,000,000 = .08\cdot(25,000,000+.5\cdot(X-2,200,000-750,000))$$
$$=1,882,000+.04\cdot X$$
$$118,000=.04\cdot X$$
$$X=2,950,000$$

End of year value is given by

$$25,000,000+2,000,000+2,950,000-2,200,000-750,000$$
$$=27,000,000$$
$$27,000,000=25,000,000\cdot(1+i)+(2,950,000-2,200,000-750,000)\cdot(1+i)^{0.5}$$
$$27,000,000=25,000,000\cdot(1+i)+0\cdot(1+i)^{0.5}$$
$$1+i=1.08$$
$$i=0.08$$

(1 mark each for X, end year value and i)

II. Fill in the blanks

1.Two factors that might influence the level of interest rates are the likelihood of <u>default</u> on payments and the possible <u>appreciation or depreciation</u> of currency.

2. The calculation of the amount of interest payable under a financial arrangement can be expressed in terms of <u>compound interest or simple interest.</u>

III. To accumulate Rs.8000 at the end of 3n years, deposits of Rs.98 are made at the end of each of the first n years and Rs.196 at the end of each of the next 2n years. The annual effective rate of interest is i . You are given $(I + i)^n = 2.0$. Determine i .

Solution:

$$98S_{3N} + 98S_{2N} = 8000$$

$$\frac{(1+i)^{3n}-1}{i} + \frac{(1+i)^{2n}-1}{i} = 81.63$$

$$\frac{8-1}{i} + \frac{4-1}{i} = 81.63$$

$10/i = 81.63$
$i = 12.25\%$

(1 mark each for the equation and final value)

IV. Big Bazaar is running a promotion during which customers have two options for payment . Option one is to pay 90% of the purchase price two months after the date of sale. Option two is to deduct X% off the purchase price and pay cash on the date of sale. A customer wishes to determine X such that she is indifferent between the two options valuing them using an effective annual interest rate of 8%.
What is the equation of value the customer would need to solve

Solution:
If we assume purchase price of 1, the present value of 90% of purchase price is 0.90 $v^{\wedge n}$. Since effective annual interest rate = 8% and n = 2 months out of 12,
$0.90\, v^{\wedge n} = 0.90 * (1/1.08)^{\wedge\, (2/12)}$

The present value should be set equal to 1-(X/100) since *X* is the percentage off the purchase price paid on date of sale and we assume purchase price of 1.

Therefore $1-(X/100) = 0.90 * (1/1.08)^{\wedge\, (2/12)}$

Dividing both sides by $v^{\wedge n}$, thereby converting the reference time point to 2 months from date of sale, we get
$(1-(X/100)) * (1.08)^{\wedge (1/6)} = 0.90$

(1 mark each for the each equation)

V. A bank offers the following choices for certificates of deposit:

Term (in years)	Nominal annual interest rate convertible quarterly
1	4.00%
3	5.00%
5	5.65%

The certificates mature at the end of the term. The bank does NOT permit early withdrawals. During the next 6 years the bank will continue to offer certificates of deposit with the same terms and interest rates.
An investor initially deposits Rs.10,000 in the bank and withdraws both principal and interest at the end of 6 years.
Calculate the maximum annual effective rate of interest the investor can earn over the 6-year period.

Solution:

There are only two real possibilities:

Two consecutive 3 year CDs:
$10000 * (1+(0.05/4))^{\wedge 12} * (1+.(0.05/4))^{\wedge 12} = 13,473.51$

One 5 year CD and a 1 year CD:
10000 * (1+(0.0565/4))^20 * (1+.(0.04/4))^4= 13,775.75
13,775.75 is the greater.

The annual effective rate is
10000 * (1+I)^6= 13,775.75

i = 5.48%

(1 mark each for each step)

[Total 13]

Q.2

I. What are the different types of loans? Describe in brief.

Solution:
The different types of loans are "interest-only" loan and repayment loan (or mortgage)
An "interest-only" loan is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.
A repayment loan is a loan that is repayable by a series of payments that include partial repayment of the loan capital in addition to the interest payments.

(I mark for naming the types, 1 mark each for the description of each type)

II. A loan is being repaid with 25 annual payments of Rs.300 each. With the 10th payment, the borrower pays an extra Rs.1000, and then repays the balance over 10 years with a revised annual payment. The effective rate of interest is 8%. Calculate the amount of the revised annual payment.

Solution:
The balance after 10 payments is
300 a_{15} @ 0.08= 300 * (8.5595) = 2567.85

The balance after an additional payment of 1000 is
2567.85 – 1000 = 1567.85

The new payment over 10 years is
1567.85 / a_{10} @ 0.08 =1567.85 / 6.7101= 233.66

(1 mark for each step)

III. An investor borrows an amount at an annual effective interest rate of 5% and will repay all interest and principal in a lump sum at the end of 10 years. She uses the amount borrowed to purchase a Rs.1000 par value 10-year bond with 8% semiannual coupons bought to yield 6% convertible semiannually. All coupon payments are reinvested at a nominal rate of 4% convertible semiannually.
Calculate the net gain to the investor at the end of 10 years after the loan is repaid.

Solution:
Price of a bond: 1000 * ((1.03)^-20+ 0.04 a $_{20\,@\,3\%}$) =1148.77=loan principal

Loan principal and interest paid: 1148.77 *(1.05)^10 =1871.23

Accumulated bond payments: 1000 * (1+ 0.04 s $_{20\,@\,2\%}$) = 1971.89

Net gain = 100.66

(1 mark for each step)

IV. A loan is repaid with level annual payments based on an annual effective interest rate of 7%.The 8th payment consists of Rs.789 of interest and Rs.211 of principal.
Calculate the amount of interest paid in the 18th payment.

Solution:
Total payment = 789 + 211 = 1000
Principal in 18th payment = Principal in 8th payment *$1.07^{(18-8)}$
Principal in 18th payment = 211 * $(1.07)^{10}$ = 415
Interest in 18th payment = 1000 – 415 = 585

(1 mark for the first 3 steps, 2 marks for the final ans)

V. Define the characteristics of government index linked bonds ?Explain in practice why most index linked securities carry some inflation risk in practice.?

Solution:
Bond issued and payments made by a government.
Coupon and redemption payments linked to an index which reflects inflation (typically lagged inflation).
Payments fixed in relation to this index, therefore bond provides inflation protection.

Most bonds have payments linked to inflation with a time lag. There is therefore a gap between the reference date for inflation used to calculate the payment from a bond and the date on which a payment is received. If inflation is higher than anticipated between those two dates, the real value of the payments will be reduced.

[Total 18]

Q.3
An institution has a liability to pay Rs.15, 000 per annum, half-yearly in arrears, forever.

(i) Calculate the present value and volatility of the liability at 8% pa effective.

(ii) Calculate the duration of the liability at 8% pa effective.

The following two stocks are available for investment:
(A) A special 5-year stock that pays a coupon of Rs.5 per Rs.100 nominal at the end of the first year rising, by 2% pa compound, to 5x 1.02^{4} at the end of the fifth year.

(B) An n-year zero-coupon bond.
The institution chooses to invest equal amounts of cash in Stock A and Stock B.

(iii) If the institution requires that the duration of the assets must equal the duration of the liabilities, show that n, the term of the zero-coupon bond, must equal 22 years if interest rates are 8% pa effective.

(iv) Do you think that the institution has managed to implement an immunisation strategy? Give reasons, but not any calculations, to support your answer.

[Total 17]

Solution:

(i) *PV and volatility*

$$P = 7,500(v^{0.5} + v + v^{1.5} + \cdots)$$

$$= 15,000a_{\overline{\infty}|}^{(2)}$$

$$= \frac{15,000}{i^{(2)}} = \frac{15,000}{2\left(1.08^{0.5} - 1\right)} = £191,178$$

The volatility, υ, equals:

$$\upsilon = -\frac{P'}{P}$$

The present value can be written in terms of i as follows:

$$P = \frac{15}{2(\sqrt{1+i} - 1)} = 7.5\left((1+i)^{0.5} - 1\right)^{-1}$$

So:

$$P' = -7.5\left((1+i)^{0.5} - 1\right)^{-2} \times 0.5(1+i)^{-0.5}$$

The volatility is then (using $i = 8\%$):

$$P' = \frac{7.5\left((1+i)^{0.5} - 1\right)^{-2} \times 0.5(1+i)^{-0.5}}{7.5\left((1+i)^{0.5} - 1\right)^{-1}} = \frac{2,344.618}{191.178} = 12.26$$

(ii) The duration (or discounted mean term) τ equals:

$$\tau = (1+i)\upsilon = 1.08 \times 12.26 = 13.25$$

(iii) ***Choosing n***

First calculate the duration of the two stocks.

Working in £100, the price of Stock A is:

$$P_A = 5\left(v + 1.02v^2 + \cdots + 1.02^2 v^5\right) + 100v^5$$

$$= \frac{5}{1.02} a_{\overline{5|}@k\%} + 100v^5$$

where $1+k = \frac{1.08}{1.02} \Rightarrow k = 5.8824\%$.

$$P_A = 4.90196 \times 4.22587 + 100 \times 0.68058 = £88.77$$

Next, calculate the duration of Stock A:

$$\tau_A = \frac{5\left(v + 2\times 1.02v^2 + 3\times 1.02^3 v^3 + \cdots + 5\times 1.02^4 v^5\right) + 5\times 100v^5}{P_A}$$

$$= \frac{\frac{5}{1.02}(Ia)_{\overline{5|}@5.8824\%} + 500v^5}{88.77}$$

$$= \frac{4.90196 \times 12.1952 + 500 \times 0.68058}{88.77}$$

$$= 4.51$$

The duration of Stock B is n years.

We require the duration of the assets to equal the duration of the liabilities. Therefore since equal amounts of cash are to be invested in each stock, we need:

$$\tfrac{1}{2}n + \tfrac{1}{2}4.51 = 13.25$$

$$\Rightarrow n = 21.99 = 22 \text{ years}$$

(iv) An immunisation strategy has not been implemented. The convexity of assets is not greater than the convexity of the liabilities because the liability cashflows are more "spread out" than the asset cashflows.

Q.4

A loan of nominal amount of Rs.100,000 is to be issued bearing interest payable quarterly in arrear at a rate of 8% p.a. Capital is to be redeemed at 105% on a coupon date between 15 and 20 years after the date of issue, inclusive, the date of redemption being at the option of the borrower.

(i) An investor who is liable to income tax at 40% and tax on capital gains at 30% wishes to purchase the entire loan at the date of issue. What price should she pay to ensure a net effective yield of at least 6% p.a.?

(ii) Exactly 10 months after issue the loan is sold to an investor who pays income tax at 20% and capital gains tax at 30%. Calculate the price this investor should pay to achieve a yield of 6% p.a. on the loan:
(a) assuming redemption at the earliest possible date
(b) assuming redemption at the latest possible date

(iii) Explain which price the investor should pay to achieve a yield of at least 6% p.a.

[Total 17]

(i) $i^{(p)} = .06^{(4)} = .058695$

$$g(1-t_1) = \frac{.08}{1.05} \times 0.6 = .045714$$

So $i^{(P)} > g(1-t_1) \Rightarrow$ there is a capital gain on the contract.

The minimum value of the loan arises if the repayment is at the latest possible date. Hence we assume redemption occurs 20 years after issue.

Let A be the price per 100 of the loan

$$A = 100 \times .08 \times 0.6 \times a_{\overline{20}|}^{(4)} + (105 - 0.3(105 - A))v^{20} \text{ at } 6\%$$

$$= 56.280 + 22.917 + 0.09354A$$

=87.370 or 87370 for the whole loan

(ii) (a) The price is A' where

$$A' = 100 \times 0.08 \times 0.8 \times \ddot{a}_{\overline{14\frac{1}{4}}|}^{(4)} \times v^{\frac{2}{12}} + 105v^{14\frac{2}{12}}$$
$$- \text{PV of Capital Gains Tax (if any)}$$

$$= 61.607 + 45.993 - \text{PV of Capital Gains Tax}$$

$$= 107.80$$

Since $A' > 105$ there is no capital gains tax liability.

(b) $A'' = 100 \times 0.08 \times .8 \times \ddot{a}^{(4)}_{\overline{19\frac{1}{4}}|} \times v^{\frac{2}{12}} + 105v^{19\frac{2}{12}} - \text{PV(CGT)} = 108.25$

(iii) Investor should pay no more than A'. If investor pays A'' and bond is redeemed early, yield achieved will be less than 6% p.a.

Q.5

I. $f_{t,r}$ is the forward rate applicable over the period t to t + r. it is the spot rate over the period 0 to t. The gross redemption yield from a one year bond with a 6% annual coupon is 6% per annum effective; the gross redemption yield from a two year bond with a 6% annual coupon is 6.3% per annum effective; and the gross redemption yield from a three year bond with a 6% annual coupon is 6.6% per annum effective. All the bonds are redeemed at par and are exactly one year from the next coupon payment.
(a) Calculate i_1, i_2 and i_3 assuming no arbitrage.

(b) Calculate $f_{0,1}$, $f_{1,1}$ and $f_{2,1}$ assuming no arbitrage.

(ii) Explain why the forward rates increase more rapidly with term than the spot rates.

(a) Clearly $i_1 = 6\%$

$$p_2 = 6(1.06)^{-1} + 106(1+i_2)^{-2}$$

$$p_2 = 6a_{\overline{2|}} + 100v^2 \text{ @ } 6.3\%$$

$$\Rightarrow (1+i_2)^{-2} = \frac{6a_{\overline{2|}6.3\%} + 100v^2_{6.3\%} - 6(1.06)^{-1}}{106}$$

$$a_{\overline{2|}6.3\%} = 1.82571 \qquad 1.06^{-1} = 0.94340$$

$$v^2_{6.3\%} = 0.88498$$

$$\Rightarrow (1+i_2)^{-2} = 0.884829 \qquad i_2 = 6.30907\%$$

$$p_3 = 6(1.06)^{-1} + 6(1.0630907)^{-2} + 106(1+i_3)^{-3}$$

$$p_3 = 6a_{\overline{3|}6.6\%} + 100v^3_{6.6\%}$$

$$\Rightarrow (1+i_3)^{-3} = \frac{6a_{\overline{3}|6.6\%} + 100v_{6.6\%}^3 - 6(1.06)^{-1} - 6(1.0630907)^{-2}}{106}$$

$a_{\overline{3}|6.6\%} = 2.643614$

$v_{6.6}^3 = 0.82552$

$1.06^{-1} = 0.94340$

$1.0630907^{-2} = 0.884829$

$\Rightarrow (1+i_3)^{-3} = 0.824947 \Rightarrow i_3 = 6.62476\%$

(b) $f_{0,1} = 6\%$

$(1+f_{1,1})(1.06) = (1.0630907)^2 \Rightarrow f_{1,1} = 6.61904\%$

$(1+f_{2,1})(1.0630907)^2 = (1.0662476)^3$

$\Rightarrow f_{2,1} = 7.25896\%$

(ii) The accumulation factors related to spot rates are geometric averages of accumulation factors related to the forward rate for the same year as well as for all those relating to all previous years. Therefore, if forward rates are increasing, spot rates, being an average of a spot rate for the given year and for previous years, will increase more slowly.

i)-(5 marks each for a) and b))
ii) 2 marks

II. You are given the following term structure of spot interest rates:

Term (in years)	Spot interest rate
1	5.00%
2	5.75%
3	6.25%
4	6.50%

A three-year annuity-immediate will be issued a year from now with annual payments of Rs.5000.Using the forward rates, calculate the present value of this annuity a year from now.

$$PV = 5000 \cdot 1.05 \cdot \left(\frac{1}{(1.0575)^2} + \frac{1}{(1.0625)^3} + \frac{1}{(1.065)^4} \right) = 13{,}152.5$$

[Total 15]

Q.6

A company is adopting a particular investment strategy such that the expected annual effective rate of return from investments is 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1 + i_t)$ is lognormally distributed where it is the return in the tth year. The company has received a premium of 1,000 and will pay the policyholder Rs.1,400 after 10 years.

(i) Calculate the expected value and standard deviation of an investment of 1,000 over 10 years, deriving all formulae that you use.

(ii) Calculate the probability that the accumulation of the investment will be less than 50% of its expected value in ten years. time.

(iii) The company has invested Rs.1,200 to meet its liability in 10 years time. Calculate the probability that it will have insufficient funds to meets its liability.

[Total 20]

(i) Let the accumulation of 1 unit over 10 years = S_{10}

$$E(S_{10}) = E[(1+i_1)(1+i_2)\ \dots\ (1+i_{10})]$$

Assuming i_t's are independent, this gives:

$$E(S_{10}) = E(1+i_1)E(1+i_2) \quad \therefore E(1+i_{10})$$

$$= 1.07 \times 1.07\ \dots\ 1.07 = 1.07^{10} = 1.96715$$

$$E(S_{10}^2) = E[(1+i_1)^2(1+i_2)^2\ \dots(1+i_{10})^2]$$

Assuming i_t's are independent, this gives:

$$E(S_{10}^2) = E(1+i_1)^2E(1+i_2)^2\ \dots\ E(1+i_{10})^2$$

$$E(1+i_t)^2 = E(1+2i_t+i_t^2)$$

$$= 1+2\times 0.07+0.07^2+0.09^2$$

$$= 1.153$$

$\therefore\ E(S_{10}^2) = 1.153^{10}$ and $\mathrm{Var}(S_{10}) = 1.153^{10} - 1.07^{20}$

$$= 0.282657$$

$\mathrm{S.D.}(S_{10}) = 0.53165$

$\Rightarrow$ Accumulation of £1,000 = £1,967.15 and standard deviation = £531.65

(ii) $1+i_t \sim LN(\mu, \sigma^2)$

$$\ln(1+i_t) \sim N(\mu, \sigma^2)$$

$$\ln((1+i_t))^{10} = \ln(1+i_t) + \dots + \ln(1+i_t) \sim N(10\mu,\ 10\sigma^2)$$

$\therefore\ (1 + i_t)^{10} \sim LN(10\mu,\ 10\sigma^2)$ let $10\mu = \mu'$ and $10\sigma^2 = \sigma'^2$

$$E(1 + i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.07$$

$$\text{Var}(1 + i_t) = \exp(2\mu + \sigma^2)\ [\exp(\sigma^2) - 1] = 0.09^2$$

$$\therefore\ \frac{0.09^2}{1.07^2} = \exp(\sigma^2) - 1 \quad \therefore\ \sigma^2 = 0.007050$$

$$\therefore\ \exp\left(\mu + \frac{0.00705}{2}\right) = 1.07 \quad \therefore\ \mu = \ln 1.07 - \frac{0.007050}{2}$$

$$= 0.064134$$

$\therefore\ \ln(S_{10}) \sim N(0.64134,\ 0.0705)$

we require probability $S_{10} < 0.983575$

$$= \text{probability that } \frac{\ln(S_{10}) - 0.64134}{\sqrt{0.0705}} < \frac{\ln 0.983575 - 0.6413}{\sqrt{0.0705}}$$

$= \Pr(Z < -2.4778)$ where $Z \sim N(0,1)$

$= 0.00661$

(iii) Require $\Pr(1{,}200(S_{10}) < 1{,}400)$

$= \Pr(S_{10} < 1.16667)$

$$= \Pr\left(\frac{\ln(S_{10}) - 0.64134}{\sqrt{0.0705}} < \frac{\ln 1.16667 - 0.64134}{\sqrt{0.0705}}\right)$$

$= \Pr(Z < -1.8349)$ where $Z \sim N(0,1)$

$= 0.03326$

****