Actuarial Society of India

Examinations

November 2006

CT4 (103 PART) – Stochastic Modelling

Indicative Solution

Question 1:

i. Kolmogrov Forward Equation: $P'(t) = P(t)\,A$ where

$$A = \begin{bmatrix} -0.1 & 0.05 & 0 & 0.05 \\ 1.0 & -1.2 & 0.1 & 0.1 \\ 0 & 0 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ii. The probability of saying in state H for 10 years is

$$\int_{10}^{\infty} 0.1e^{-0.1x}\,dx = e^{-1}$$

iii. First transition from H must be to S or D, each equally likely. If to D, then it is certain that no terminal illness will occur; otherwise probability of avoiding a terminal illness is d_S.

From S similarly, except that the transition probabilities are to H with probability (1.0/1.2) = (5/6), to D or T each with probability (0.1/1.2) = (1/12). Once in T it is not possible to avoid terminal illness.

Solving the above equations,

$$d_S = \frac{1}{12} + \frac{5}{6} * \frac{1}{2}\left(1 + d_S\right)$$

$$\Rightarrow d_S = \frac{6}{7} \quad and \quad d_H = \frac{13}{14}$$

iv. The Markov property implies that the time spent in state T has exponential distribution. The rate is 0.4 per year, so the expectation is 2.5 years.

The expected time spent in terminal illness given current health is

$$P\left(ever\ hit\ T \mid X_0 = H\right) * 2.5\ years = \frac{2.5}{14}\ years.$$

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Question 2:

i. Let X_n be the number of different cards that Anil has after he has bought n éclairs packets. Then since each card is equally likely in the new packet, $\{X_n: n = 0, 1, 2, \ldots\}$ will be a Markov chain with state space {0, 1, 2, …., 50} and transition probabilities such that :

$$(X_{n+1} \mid X_n = x) = \begin{cases} x & \text{with probability } \dfrac{x}{50} \\ x+1 & \text{with probability } 1-\dfrac{x}{50} \end{cases}$$

ii. The expected number of packets that Anil needs to buy to get a complete set is

$$E_{50} = \left(\frac{50}{50}\right)^{-1} + \left(\frac{49}{50}\right)^{-1} + \left(\frac{48}{50}\right)^{-1} + \ldots\ldots + \left(\frac{1}{50}\right)^{-1}$$

$$= \left(\frac{50}{50}\right) + \left(\frac{50}{49}\right) + \left(\frac{50}{48}\right) + \ldots\ldots + \left(\frac{50}{1}\right) = 50\sum_{k=1}^{50}\frac{1}{k} \approx 50(\ln 50 + 0.5771) = 224.5$$

iii. The transition probabilities would now be

$$(X_{n+1} \mid X_n = x) = \begin{cases} x & \text{with probability } \left(\dfrac{x}{50}\right)\left(\dfrac{x-1}{49}\right) \\ x+1 & \text{with probability } 2\left(\dfrac{x}{50}\right)\left(\dfrac{50-x}{49}\right) \\ x+2 & \text{with probability } \left(\dfrac{50-x}{50}\right)\left(\dfrac{49-x}{49}\right) \end{cases}$$

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Question 3:

i. Consider the following:

	Level at the start of this year after:			
Level at start of previous year	*0 claims in the previous year*	*1 claim in the previous year*	*2 claims in the previous year*	*3 or more claims in the previous year*
5	4	5	5	5
4	3	5	5	5
3	2	4	5	5
2	1	3	4	5
1	1	1	2	5

For each policyholder, the number of claims in each year has a Poisson (0.25) distribution. So,

P(0 claims)=	$e^{-0.25}$	*= 0.7788*
P(1 claim)=	$0.25\, e^{-0.25}$	*= 0.1947*
P(2 claims)=	$(0.25^2 * e^{-0.25})/2$	*= 0.0243*
P(3 or more claims)=	*1- 0.7788 – 0.1947 – 0.0243*	*= 0.0022*

Thus, the transition probability matrix is:

$$P = \begin{bmatrix} 0.9735 & 0.0243 & 0 & 0 & 0.0022 \\ 0.7788 & 0 & 0.1947 & 0.0243 & 0.0022 \\ 0 & 0.7788 & 0 & 0.1947 & 0.0265 \\ 0 & 0 & 0.7788 & 0 & 0.2212 \\ 0 & 0 & 0 & 0.7788 & 0.2212 \end{bmatrix}$$

ii. In order to be in Level 1 in year 3, the policyholder requires two consecutive claim-free years. The probability of this is $0.7788^2 = 0.6065$.

A similar argument can be used for probability in Level 3 in year 3, but it may be simpler to calculate the whole vector of probabilities x_3, where

$$x_1 = (0 \quad 0 \quad 1 \quad 0 \quad 0)$$
$$x_2 = (0 \quad 0 \quad 1 \quad 0 \quad 0)P = (0 \quad 0.7788 \quad 0 \quad 0.1947 \quad 0.0265)$$
$$x_3 = (0 \quad 0.7788 \quad 0 \quad 0.1947 \quad 0.0265)P = (0.6065 \quad 0 \quad 0.3033 \quad 0.0396 \quad 0.0506)$$

Probability of being in level 3 is 0.3033 or 30.33%.

iii. Consider the following:

a. The required conditions are that the chain is irreducible and aperiodic.

b. Irreducibility: *Level I* can be reached from *level j* in **|*j – I*|** steps
Aperiodicty: $p_{ii} > 0$ *for some I*

c. The stationary distribution π will not depend on the starting position. We require:

$$(\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \pi_5)P = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \pi_5)$$

This gives the following equations:

$$0.9735\pi_1 + 0.7788\pi_2 = \pi_1 \quad (1)$$

$$0.0243\pi_1 + 0.7788\pi_3 = \pi_2 \quad (2)$$

$$0.1947\pi_2 + 0.7788\pi_4 = \pi_3 \quad (3)$$

$$0.0243\pi_2 + 0.1947\pi_3 + 0.7788\pi_5 = \pi_4 \quad (4)$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1 \quad (5)$$

Solving these simultaneous equations,

$$from\ (1) \Rightarrow \pi_2 - \frac{1-0.9735}{0.7788}\pi_1 = 0.0340\pi_1$$

$$substitute\ \ for\ \pi_2\ in\ (2) \Rightarrow \pi_3 = \frac{0.0340-0.0243}{0.7788}\pi_1 = 0.01244\pi_1$$

$$substitute\ \ for\ \pi_2\ and\ \pi_3\ in\ (3) \Rightarrow \pi_4 = \frac{0.01244-0.1947*0.0340}{0.7788}\pi_1 = 0.00747\pi_1$$

$$substitute\ \ for\ \pi_2, \pi_3\ and\ \pi_4\ in\ (4) \Rightarrow \pi_5 = \frac{0.00747-0.1947*0.01244-0.0243*0.0340}{0.7788}\pi_1 = 0.00541\pi_1$$

$$and\ substituting\ \ for\ \pi_2, \pi_3, \pi_4\ and\ \pi_5\ in\ (5) \Rightarrow \pi_1 = 0.9440$$

$$\Rightarrow \pi_2 = 0.0321,\ \pi_3 = 0.0117,\ \pi_4 = 0.0071\ and\ \pi_5 = 0.0051,$$

iv. A chi – squared goodness of fit test is best here.

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[Total 50]