Institute of Actuaries of India

Subject ST6 – Finance & Investment B

October/November 2007 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1.

a. The relationship between the futures price F_t and the spot price S_t at time t is $F_t = S_t e^{(r-r_f)(T-t)}$ since the interest rates are non stochastic.

Suppose that the hedge ratio is h. The price obtained with hedging is $h(F_0 - F_t) + S_t$ since it would sell the foreign currency in futures at the initial price of F_0 and at time t will close the contract.

Substituting F_t we get $hF_0 + S_t - hS_t e^{(r-r_f)(T-t)}$,which will reduce to hF_0 and a zero variance hedge is obtained when $h = e^{-(r-r_f)(T-t)}$.

b. When t is one day h is approximately $e^{(r_f-r)(T)} = \dfrac{S_0}{F_0}$ This gives the appropriate hedge ratio.

[Total 4]

2) a) Lets calculate the cost of borrowing in gold in monetary terms. Borrow 1 ounce of gold and buy 1.03 ounce of gold using forward contract to repay the loan. Forward price of gold will be $S_0 e^{8\%+0.5\%}$ = 1.08872 S_0 But we need to repay 1.03 ounce and hence the effective cost of borrowing will be 12.14%.

This seems lower than the currency borrowing rate of 13%.

It assumes that the company will be able to buy the forward contract at the calculated rate. The market rate may be different then the above calculated rate.

It is also possible that the markets do not offer any arbitrage opportunities but the bid offer spreads are quite large. Please note that we have not used the offer price for the forward contract which may be higher than the theoretical price calculated above.

From the perspective of Infosys, it may have to provide some comfort to the counterparty for entering into the forward contract because certainly it has a high credit risk as can be gauged from the spread between 13% and 8.32%. This may increase the cost of borrowing for Infosys.

From the perspective of the bank, it seems that it has not priced the credit risk fully while offering the gold loan.

The bank may do this if it has an offsetting transaction already on a book and it is saving some transaction cost by not off loading the transaction in the inter-bank market.

It is also possible that the debt market, gold loan market and forward markets are not arbitrage free which is prevented by regulations and hence allows Infosys to lower its borrowing costs.

[Max 6]

b) The change in S during the first three years has the probability distribution Normal(3*2,3*3^0.5)

The change during the next 3 years has the probability distribution Normal(3*3,3*6^0.5).

The change during the six years is sum of a variable with probability distribution Normal(6,3*3^0.5) and a variable with probability distribution Normal(9, 3*6^0.5). The probability distribution of the change is therefore Normal(9+6, sqrt(3^2*3+3^2*6)) = Normal(15, 9)

Since, the initial value of the variable is 10, the probability distribution of the value of the variable at the end of the year six is Normal(25, 9)

[Total 10]

3 a) Let $G(S,t)=h(t,T)S_t^n$ then $\partial G/\partial t = h_t S_t^n$, $\partial G/\partial S = hnS_t^{n-1}$ and $\partial^2 G/\partial S^2 = hn(n-1)S_t^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black Scholes differential equation we get

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

b) The derivative is worth S^n when t=T. The boundary condition for this differential equation is therefore $h(T,T)=1$.

c) The equation simplifies to $\dfrac{h_t}{h} = -rn - \dfrac{1}{2}\sigma^2 n(n-1) + r$

or integrating both sides give,

$\ln(h) = -\left\{ r(n-1) + \frac{1}{2}\sigma^2 n(n-1) \right\} t + k$ where k is a constant.

Since h=1 i.e. ln(h)=0 when t=T gives the value of k = $\left\{ r(n-1) + \frac{1}{2}\sigma^2 n(n-1) \right\} T$.

Therefore, $\ln(h) = \left\{ r(n-1) + \frac{1}{2}\sigma^2 n(n-1) \right\}(T-t)$ or $h(t,T) = e^{\left\{ r(n-1)+\frac{1}{2}\sigma^2 n(n-1) \right\}(T-t)}$

[Total 8]

4 a) Given a numeraire Bt and a tradable asset St, a process Vt represents a tradable asset if and only if its discounted value Bt-1Vt is actually a Q-martingale, where Q is the measure under which the discounted asset Bt-1St is a martingale.

b) The differential equation for St can be derived using Ito's formula and is given by $dS = rSdt + \sigma SdW$. Now $B_t^{-1}S_t$ is martingale under Q because St is tradable. The differential equation for $B_t^{-1}S_t$ can be derived as follows $d(B_t^{-1}S_t) = B_t^{-1}d(S_t) + S_t d(B_t^{-1})$ or $= B_t^{-1}(rS_t dt + \sigma S_t dW_t) - rS_t B_t^{-1}dt = B_t^{-1}\sigma S_t dW_t$. Since this is Q-martingale means W is Brownian motion under Q.

Now, let's find out the differential equation for S_t^θ where θ is a constant. Again using Ito's lemma we get, $d(S^\theta) = \sigma S\theta S^{\theta-1}dW + (\theta S^{\theta-1}rS + \frac{1}{2}\theta(\theta-1)S^{\theta-2}\sigma^2 S^2)dt$ or

$$= \sigma\theta S^\theta dW + (\theta S^\theta r + \frac{1}{2}\theta(\theta-1)S^\theta\sigma^2)dt .$$

Now we can substitute this to calculate the differential for $B_t^{-1}S^\theta$ which is derived as follows

$$d(B_t^{-1}S^\theta) = B_t^{-1}d(S^\theta) + S^\theta d(B_t^{-1}) \quad \text{or}$$

$$= B_t^{-1}\sigma\theta S^\theta dW + B_t^{-1}(\theta S^\theta r + \frac{1}{2}\theta(\theta-1)S^\theta\sigma^2 - rS^\theta)dt \quad \text{or}$$

$$= B_t^{-1}\sigma\theta S^\theta dW + B_t^{-1}S^\theta(\theta r + \frac{1}{2}\theta(\theta-1)\sigma^2 - r)dt.$$

Under measure Q, W is Brownian and for $B_t^{-1}S^\theta$ to be martingale under Q the drift term should be zero.

Substituting the value of 2 for θ we get $B_t^{-1}S^2(2r+\sigma^2-r)$, a non zero value and hence S_t^2 is not tradable.

Substituting the value of $-\frac{2r}{\sigma^2}$ for θ we get, $B_t^{-1}S^{-2r/\sigma^2}(-\frac{2r^2}{\sigma^2} - r(-\frac{2r}{\sigma^2}-1) - r)$ which is equal to zero. Therefore, S^{-2r/σ^2} is a tradable asset.

[Total 10]

5 (a). **<u>Graph to be drawn</u>**

(b). Payments to Ram & Co (5% cap)

The benchmark rate did not exceed 5.5% on 1st January, 1st April or 1st July. So there will have been no payments on 1st April, 1st July or 1st October.

On 1st January the following year Ram & Co will receive:

100 X 0.25 X (6.0 – 5.5) = 12.5

based on the rate applicable on 1st October

Payments to Gopal & Co (5% floor)

The benchmark rate was below 5.5% on 1st January, 1st April and 1st July. So Gopal & Co., will receive

On 1st April: 100 X 0.25 X (5.5 – 5) = 12.5
(based on the rate applicable on 1st January)

Normally, no payment is made on the fist reset date under a cap or a floor. However, There will be a payment here on April 1, as we are told in the question that the floor was taken out in the previous year.

On 1 July: 100 X 0.25x(5.5 – 5.25) = 6.25
(based on the rate applicable on 1 April)

On 1 October: 100X0.25X(5.5 -5.25)= 6.25
(based on the rate applicable on 1 July)

The rate applicable on 1st October was above 5.5% and so on payment will be made on 1st January the following year.

5 (c). Price of the Swap

If the market is arbitrage-free, the prices of caps, floors and "pay-fixed-for-floating" interest rate swaps will be connected via the put-call parity relationship:

cap price = floor price + swap price

This is because a long holding of 1 cap and a short holding of 1 floor will reproduce the cashflows from the swap.

Note that the "price" of a swap can be positive or negative, depending on how the strike price (=5.5% here) compares with market expectations.

So, rearranging this, we have:

Swap price = cap price – floor price

5 (d). Price of the collar
A portfolio consisting of +1 cap (with strike price 6%) and -1 floor (with strike price 4%) will reproduce the cashflows from the collar

So (if the market is arbitrage – free and ignoring transactions costs) the price of the collar will be the difference 0.8 % - 0.6% = 0.2%

[Total 10]

6. (a). The rationale for a term structure yield curve model is to be able to price simultaneously options spread across the entire range of maturities in a yield curve. Exotic interest rate swaps and options include spread options, Bermudan swaptions and path-dependent options (knock-outs etc). These depends not just on the evolution of forward rates along the curve, but on the correlation between changes.

Desirable features are:

- The current swap (or bond) curve should be reproduced by the model
- Easy to specify and calculate (on a suitable computer)
- Easy to calibrate – for example, a long-normal expression of the model will help fit to cap prices, which are traded based on the standard long-normal Black model

- Enough degrees of freedom (parameters) to make the model flexible to cope with any yield curve shape, but not overly flexible so there is instability between parameters from one day to the next
- Volatility of rates of different maturity should be different, with shorter rates usually being more volatile.
- Imperfect correlation between forward rates, although this is only needed when pricing certain types of options where correlation has a big effect on the price (e.g. yield spread options, callable swaptions)
- Interest rtes cannot be negative
- Reasonable dispersion of rates over time 9 due to Brownian motion) – too large a probability of getting hugely high or low values will distort the model
- One possible way of allowing for this is to make the model " mean-reverting", that is, when rates go too high (or low), they tend to revert back to some central level

(b).

i. <u>One-factor equilibrium models</u>

A one-factor model creates a process for a variable, usually the short rate ***r***, which leads to an evolution of rates over time.

Parameters (which can be time-and curve-dependent) govern the evolution. From this evolution, all present and future bonds and swap rates can be priced.

Single factor models are usually easy to use and calibrate, and can even in some simple cases lead to analytical solutions.

Equilibrium models are a particular class of models where a simplified form of an entire economy is described.

All securities and contingent claims are priced endogenously in equilibrium models. This gives a world of "absolute" prices, which may differ from real market prices.

In practice, the simplifications of the economy are so great that the resulting yield curve shares are limited. This can lead to inaccurate pricing of securities.

ii. <u>No-arbitrage models.</u>

No-arbitrage, or "arbitrage free", models are a class of models which allow recovery of market prices of one set of securities given prices of another set. This creates a world of "relative" prices.

In a non-"arbitrage free" model, securities could be priced using the model and then traded at a different price in the real world, leading to persistent profit. In simplest terms, no-arbitrage is the absence of a "free lunch".

No-arbitrage is very important in yield curve models, since most complex structures are limiting case of simpler structures (such as swaps, caps, floors) and ideally the model should recover the prices of the latter exactly.

Also, hedging is done using the simpler structures, so the accounting process will not be distorted by imaginary gains and losses.

[Total 10]

7. (a). Continuously compounded rate:

5 year zero coupon bond has rate of interest 4.45% pa.

Force of interest = $\log_e$ (1.0445) [$\delta=\log_e(l+i)$] = 4.354%

(b). A 3 year zcb has rate of 3.9% pa

A 4 year zcb has rate of 4.2 % pa

Using linear interpolation (simple mean) a 3 ½ year zcb has rate of 4.05 % pa (alternatively, geometric mean of the rates may also be taken, which would come to 4.0499% ≈ 4.05%)

Value per Re. 1 nominal would be

$(1.045)^{-3.5}$ = 0.8572 = re. 0.86 per Re. 1 nominal

©. Let x be the value of interest per Rs. 100,

value of a 5 year zcb now $\dfrac{100}{1.0445^5} = 80.44$

$$x[1.035^{-1} + 1.036^{-2} + 1.039^{-3} + 1.042^{-4} + 1.0445^{-5}] + 100 \times 1.0445^{-5} = 100$$

$$\Rightarrow x[0.96618 + 0.93171 + 0.89157 + 0.84826 + 0.80437] + 80.44 = 100$$

$$\Rightarrow x = \frac{19.56}{4.44209} = 4.403 on Rs. 100 no\min al value$$

∴ Flat coupon rate = 4.4 % of nominal value.

(d).

$$= (1.036)^2 \times \left[1.039^{-3} + 1.042^{-4} + 1.0445^{-5} + 1.0460^{-6} + 1.0475^{-1}\right] + (1.036)^2 \times 100 \times 1.0475^{-7} = 100$$
$$\Rightarrow a = 1.036^2 \times \left[0.89157 + 0.84826 + 0.80437 + 0.76351 + 0.72264\right] + 1.036^2 \times 100 \times 0.72264 = 100$$
$$\Rightarrow 4.32575x = 22.43934$$
$$X = 5.1874\, perRs.100$$

Coupon rate = 5.187% on nominal value

(e) The fixed leg coupon is same as above.

i). 4.4%. This is because first year fixed coupon shall be the same as that on 5-year par value annual interest rate swap.

This is because this payment that is made at the end of year 1 is reset now i.e. at time 0, on which date we know the one year forward rate.

ii). In this case, the second payment is made at end of year 2 based on the rate at the end of year 1 i.e. at beginning of year 2. However, based on zcb yields, we can calculate the 1 year forward rate at time 1. This is just a forward rate for entering into a contract now. However, in case of an interest rate swap, a contract is "re-entered" on every reset date. Therefore it is essential that we estimate the forward rate as what it would be on that reset date. Assuming interest rates are log normally distributed, we can estimate the parameters μ and σ based on method of moments (by studying the behaviour of forward rates during the past 5 or 6 years say) and then calculate confidence intervals of the forward rate. This would give us a range of probable range of one year forward rates, given the zero coupon bond yield. Hence, we need further information as regards:

→ mean of one year forward rates
→ the standard deviation

[Total 10]

8. a. (i) Define Kappa (vega): Kappa (vega) is the sensitivity of the price of a derivative to small changes in the assume volatility of the underlying asset (assuming the other parameter values are unchanged). Mathematically:

$$K = \frac{\partial f}{\partial \sigma}$$

(ii). Derive a formula for kappa (vega)
According to the Garman – Kohlhagen model, the price of a call option on a dividend-paying share is

$c_t = S_t e^{-q(T-t)}\Phi(d_{1)} - Ke^{-r(T-t)}\Phi(d_2)$

Differentiating with respect to σ gives:

$$k = \frac{\partial c_t}{\partial \sigma} = s_t e^{-q(T-t)}\phi(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial \sigma}$$

Applying the lemma, we get:

$$K = S_t e^{-q(T-t)}\phi(d_1)\frac{\partial d_1}{\partial \sigma} - S_t e^{-q(T-t)}\phi(d_1)\frac{\partial d_2}{\partial \sigma}$$
$$= S_t e^{-q(T-t)}\phi(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right]$$
$$= S_t e^{-q(T-t)}\phi(d_1)\frac{\partial}{\partial \sigma}(d_1 - d_2)$$

But we know (again from the Tables) that $d_2 = d_1 - \sigma\sqrt{T-t}$. So we get:

$$k = S_t e^{-q(T-t)}\phi(d_1)\frac{\partial}{\partial \sigma}(\sigma\sqrt{T-t})$$
$$= S_t e^{-q(T-t)}\phi(d_1)\sqrt{T-t}$$

b.Evaluate kappa (vega)

We can apply the formula part (ii) to find the kapps (vegas) for the call options:

(i). $K_a = 212e^{-0.04(90/360)}\phi(0.480)\sqrt{90/360} = 37$

(ii). $K_b = 250e^{-0.04(90/360)}\phi(-1.007)\sqrt{90/360} = 25$

(iii). $K_c = 212e^{-0.04(15/360)}\phi(0.989)\sqrt{15/360} = 11$

For the put option, we can use the put-call parity relationship (from page 47 of the Tables). This tells us that:

$$c_t + S_t e^{-q(T-t)} = p_t + Ke^{-r(T-t)}$$

So, differentiating with respect to σ, we see that:

$$\frac{\partial c_t}{\partial \sigma} = \frac{\partial p_t}{\partial \sigma}$$

ie the kappas (vegas) for calls and puts are the same.

So we get:

(iv) $k_d = k_a = 37$

(c). Comment

i. Kappa (vega) is always positive, since increasing the assumed volatility increases the probability that the price of the underlying asset will move in the holder's favour, but the option protects the holder from losses resulting from an adverse price movement.
ii. Options that are close to being at-the-money will have a higher kappa, since the volatility directly affects the time value (which is greater than). So $K_a > K_b$.
iii. Options with longer remaining terms have more scope for movement in the price of the underlying. So $K_a > K_c$
iv. As noted above, kappa is the same for puts and calls. So $K_d = K_a$

[Total 10]

9. (a). Fund = 275 crore, Index = 1,100 . Therefore fund = 250 times the index.
Fund falls 5 % iff index alls 5 % (to 1045).
So puts of 250 times the index with strike 1,045 are required
Apply Black Scholes with
S_0 = 1,100; K = 1,045;p T= 1
R = 5% (risk free rate): σ = 25%; y = 3% (dividend yield)

Which leads to
[Note: in the following formula, candidates may equally calculate S_0 exp (-yT) and take it inside the logarithm instead of having an explicit term in y.]

$$d = \frac{\ln(S_0 / K) + (r - y + \sigma^2 / 2)T}{\sigma\sqrt{T}} = 0.41017$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.16017$$
$$\Phi(-d_1) = 0.34084$$
$$\Phi(-d_2) = 0.43657$$

Value of one put = K exp (-rT)$\Phi(-d_2)$-S_0exp(-yT)$\Phi(-d_1)$ = 69.9236

Total cost of insurance = 250 X 69.9236 = 17480.9

(b) From put-call parity (or by re-arranging the Black Scholes equation for valuing puts and calls) we have

$$V_{put} = V_{call} - S_o e^{yT} + Ke^{-rT}$$

This shows that a put option can be created by shorting e^{-qT} X the index, buying a call option (of same term and strike as the put) and investing the remainder in the risk free asset.

For fund manager under question, this involves:
Selling 275 croreXexp(-yT) = 266,872, 500 of stock.

Investing in 250 crore X the S&P-500 index of calls with strike 1,045 and Term 1 year.

Investing the remainder in 1 year risk free zero coupon bonds yielding 5%

(c). The delta of one put option is

$e^{-yT}(\Phi(d_1)-1) = e^{-0.03*1}(0.65916-1) = -0.33077$

Hence 33.077% of the portfolio (i.e. 90.962 crore) needs to be sold (shorted) and the proceeds invested in risk free assets.

(d). The delta of a nine month index futures contract is

$e^{(r-y)T} = e^{(5\%-3\%)x0.75} = 1.0151.$

The short position required is $\frac{90.962}{1,110}$ = 82,693 times the index.

Hence a short position involving $\frac{82693}{1.0151*250}$ = 325.85 (which rounds to 326) futures contracts is needed.

[Total 10]

10. (a). (i). Caplet prices: final payoff at node $j = \max\{r_j(t) - 3.5, 0\}$

t = 1:

	0.2429
0.1178	
	0.0000

t= 2:

		1.5523
	0.7481	
0.3629		0.0000
	0.0000	
		0.0000

t= 3:

2.9945

		1.8138	
	1.0653		0.8193
0.6107		0.3967	
	0.1937		0.0000
		0.0000	
			0.0000

Floorlet prices: final payoff at node *J* = max {3.5 - r_j(t), 0}

t = 1:

	0.0000
0.5135	
	1.0584

t = 2:

		0.0000
	0.1064	
0.4319		0.2208
	0.7840	
		1.3851

t=3:

			0.0000
		0.0000	
	0.1513		0.0000
0.4118		0.3140	
	0.6975		0.6486
		1.1146	
			1.6270

Hence cap price = 0.1178+0.3629+0.6107 = 1.9105 and floor price = 0.5135 + 0.4319 + 0.4118 = 1.3572. These prices are in % nominal.

Delta for any caplet or floorlet is $(P_{up} - P_{down})/(r_{up} - r_{down})$ *at t= 0. Given that* $r_{up} - r_{down}$ *= 3.752-2.416 = 1.336, we get:*

Caplet Δ for *t* = 1: 0.1818 t=2: 0.5600 r=3: 0.6524

Floorlet Δ for t = 1: -0.7922 t=2: -0.5072 t=3: -0.4088

These are all separate options, so we don't add the deltas.

The numerical values were not required by the examiners; they are given to assist in the understanding of the solution.

<u>Check put-call parity of prices</u>

If we take any caplet price minus the floorlet price, the final payoff is r(t) -3.5 at each node.

Hence, since the tree creates the present value of the final payoff, the value at t = 0 will be the present value of the forward rate less the present value of the strike, which is put-call parity.

Numerical examples could have been given instead to show Put-Call parity for full marks.

(b). Comparison with Black model
Both BDT and Black use a log-normal model so using the caplet volatilities for each time step should give very close answers.

BDT should give exact same answers as Black for caps and floors if properly calibrated and small Δt (i.e. $\Delta t \rightarrow 0$).

Our BDT tree is very coarse, so won't get very precise answers (though amazingly they are very close). Need a smaller time step to be accurage.

Need to be sure day-count methods are same.

Generally our deltas will be poor due to the large time step.

Black deltas are always w.r.t. the forward rate, not the short rate as we calculated.

BDT deltas can be converted into forward rate deltas by first calculating, then dividing by, the sensitivity of the forward rate to the short rate as we calculated.

BDT deltas can be converted into forward rate deltas by first calculating, then dividing by, the sensitivity of the forward rate to the short rate but this is a messy calculation and needs a small time step.

(c). 2-year bond option

Price the bond at each node at the end of year 2 using the discount process in (*). Remember to allow for coupons due after year 2 but not before (the option will be based on the clean price).

Using these prices, calculate the final payoff at each node of t = 2 and value the option in the same way as the caplets above using (*).

The answer will be suspect, as the option life and bond life are too similar.

In reality, the correlation of rates will affect the value of the option (in general, BDT will overvalue). BDT does not have an explicit way of expressing the imperfect correlation of forward rate.

[Total 10]

11. (a). A bull spread provides a call at lower price than a single option, but with limited upside and no downside except the premium difference.

A fund manager might buy equity bull spreads at-the-money on the lower strike and out-of-the-money on the higher. This would enable him/her to enter the market for a limited risk but reduce the cost of entry.

b & c to be drawn **(2+2 = 4)**

(d). (i). IF volatility falls, the time value will be lower, so the value will approach the expiry value….
….. which implies that above ~ 250 the value will increase, below ~250 it will decrease.

(ii). If time to expiry decrease, this has the same effect as (a), although the exact amplitudes will be different.

[Total 8]

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