Institute of Actuaries of India

Subject CT8-Financial Economics

October/Nov 2007 Examination

INDICATIVE SOLUTION

INSTRUCTIONS TO MARKERS

1. MARKS ARE ALLOTTED FOR EACH BIT OF QUESTION IN THE MAIN EXAM PAPER, AS WELL AS IN THE MODEL/INDICATED SOLUTIONS.
2. MARKERS ARE REQUESTED TO ALLOT MARKS AS PER THE SCHEME INDICATED BELOW.
3. PLEASE USE AN EXCEL SHEET (SOFTCOPY) WHEN MARKER'S SHEET IS FURNISHED TO IOAI—[SO THAT TOTALLING MISTAKES DO NOT TAKE PLACE, AND ALSO FOR FURTHER EVALUATION/RESEARCH OF SCRIPTS AT IOAI'S OFFICE AND REVIEW EXAMINER]

Q1

(1) Inflation within the Wilkie Model

- Wilkie Model has a hierarchical structure in which inflation is a key higher-order variable that is assumed to influence other variables within the model but is itself not influenced by the other variables**.**
- Wilkie models the continuously compounded force of inflation as:

  $I(t)=\ln Q(t)-\ln Q(t-1)$

  Where Q(t) is the CPI(consumer price index)

  - Some variables may be explicitly dependant on inflation eg

  K(t)---force of dividend growth during year t

  Implicitly related to inflation:

  K(t) again through DM(t), an exponentially –weighted average of past inflation

  $DM(t)=DD. \quad (1-DD)^k I(t-k)$

Total---3 mark

(2)(a) **Expression for long-term average inflation rate**

$E[I(t)] = E[a + bI(t-1) + e(t)] = a + b E[I(t)] + 0$

i.e long-term avg interest rate is $E[I(t)]=a/(1-b)$

(2)(b**) Economic justification for AR(1) process:**

-Major determinant of price inflation is wage inflation

-Wages are usually negotiated annually and any increases in a year are reflective of the increase in the general price level over the previous year.Any dependence on previous years is likely to be much weaker---i.e AR(1) process

-- $-1<b<1 \Rightarrow$ mean-reverting, as inflation usually is. Higher than average levels of inflation in one year are more likely to be followed by a reduction in inflation in the following year, especially if authorities have an inflation target.Also, inflation is normally a stationary process which does not increase or decrease without bound.

(2)(c)**Why model is unsuitable for share prices**

--mean-reverting AR(1) process is unsuitable for share prices are not stationary or mean-reverting. They usually increase without bound i.e. non-stationary
--Unsuitable if share markets are believed to be weak-form efficient
--Share price increments are not lognormally distributed, as assumed here i.e prices can jump, crash & change very little also---not consistent with normal distribution.

Q2

SOLUTION Q2

(i) TST P(t,T) is a solution of

$$\frac{\partial P}{\partial t} + 0.5\mathrm{r}\sigma^2 \frac{\partial^2 P}{\partial r_t^{\,2}} - rP + a(\mu - r_t)\frac{\partial P}{\partial r_t} = 0 \quad \text{-----(1)}$$

Substitute P(t,T) in eqn(1) to get

$$\frac{\partial P}{\partial t} = \left[\frac{\partial A}{\partial t} - r_t \frac{\partial B}{\partial t}\right]$$

$$\frac{1}{2} r_t \sigma^2 \frac{\partial^2 P}{\partial t^2} = \frac{1}{2} r_t \sigma^2 B^2 P$$

$$\mathrm{a}((\mu - r_t)\frac{\partial P}{\partial r_t} = a(\mu - r_t)(-1)PB$$

Substituting these values in eqn(1):

$$\mathrm{P(t,T)}\left[\left\{\frac{\partial A}{\partial t} - r_t \frac{\partial B}{\partial t}\right\} + 0.5 r_t \sigma^2 B^2 - r_t - a(\mu - r_t)B\right] = 0$$

$$\mathrm{Or,P(t,T)}\left[\left\{\frac{\partial A}{\partial t} - a\mu B\right\} + r_t\left\{-\frac{\partial B}{\partial t} + \frac{1}{2}\sigma^2 B^2 - 1 + aB\right\}\right] = 0$$

This is possible if

$$\left[\left\{\frac{\partial A}{\partial t} - a\mu B\right\}\right] = 0 \text{ -----(2)}$$

$$\left[\left\{-\frac{\partial B}{\partial t}+\frac{1}{2}\sigma^2 B^2 -1+aB\right\}\right]=0-----(3)---------$$

(since r_t, P(t, T)≠0)

Now,

$$\frac{\partial A}{\partial t}=\frac{2(9e^{0.25(T-t)}+1)}{10e^{0.225(T-t)}}\left[\left(\left(9e^{0.25(T-t)}+1\right)(-2.25e^{0.225(T-t)})-10e^{0.225(T-t)}(-2.25e^{0.25(T-t)})\right)\right]/(9e^{0.25(T-t)}+1)^2$$

$$= \quad \frac{2.25\times 2(e^{0.25(T-t)}-1)}{10(9e^{0.25(T-t)}+1)}$$

=0.45B/40=0.1125B

From eqn (2), ⇒aμ=0.1125-----(4)

$$\frac{\partial B}{\partial t}=40\left[(e^{0.25(T-t)}(-1)0.25)(9e^{0.25(T-t)}+1)-(e^{0.25(T-t)}-1)(-9\times 0.25e^{0.25(T-t))}\right]/(9e^{0.25(T-t)}+1)^2$$

$$=-100\frac{e^{0.25(T-t)}}{(9e^{0.25(T-t)}+1)^2} \quad -----(5)$$

$$\frac{1}{2}\sigma^2 B^2 =800\sigma^2\left[\frac{e^{0.5(T-t)}+1-2e^{0.25(T-t)}}{(9e^{0.25(T-t)}+1)^2}\right]$$

$$aB=40a\frac{\left[9e^{0.5(T-t)}-8e^{0.25(T-t)}-1\right]}{\left[9e^{0.25(T-t)}+1\right]^2}$$

From $\left[\left\{-\frac{\partial B}{\partial t}+\frac{1}{2}\sigma^2 B^2 -1+aB\right\}\right]=0$

By substituting the above terms, we get

$$\left[e^{0.25(T-t)}(100-1600\sigma^2-320a)+e^{0.5(T-t)}(800\sigma^2+360a)+(800\sigma^2-40a)\right]=81e^{0.5(T-t)}+18e^{0.25(T-t)}$$

Equating the coefficients of various factors:

$800\sigma^2+360a=81$ -----(6)
$100-1600\sigma^2-320a=18$ -----(7)
$800\sigma^2-40a=1$ -----(8)

(Total 9 marks)

Values of a, u, sigma which determine the solution follow in next part

(ii)Solving the equations (6), (8) gives

400a=80, or a=0.2(Ans)

From eqn (4), $a\mu=0.1125$

$\therefore \mu=0.05625$(Ans)

From eqn(8)

$800\sigma^2=1+40*0.2=9 \quad \therefore \sigma=0.1061$(Ans) -----

(Total 3 marks)

(iii)

$$R(0,T) = -\ln P(0,T)/T$$

$$= \{B(0,T)r_t - A(0,T)\}/T$$

$$\underset{T\to\infty}{Limit}\, B(0,T) = 40/9$$

$$\underset{T\to\infty}{Limit}\, A(0,T) = \underset{T\to\infty}{Limit}\, 2\ln\left[\frac{10e^{0.225T}}{9e^{0.25T}+1}\right]$$

$$\cong 2\ln\left[\frac{10e^{-0.025T}}{9}\right]$$

$$= 2\ln 10/9 - 0.025T * 2$$

$$\underset{T\to\infty}{Limit}\, R(0,T) = 0.05(Ans)$$

Q3 (8 marks)

SOLUTION

i. Assume the current time is ‘t’, the time of expiry is T,’q’ is the dividend yield, risk-free rate is ‘r’ and the investor holds the underlying share currently worth S_t.

On early exercise, the exercise price E can be invested to give E exp[r(T-t)] at T.
By waiting until T, the investor can then obtain Max(E, S_T) + qS_t exp[r(T-t)].

Hence share should definitely **not** be exercised early if :

E+ qS_t exp[r(T-t)].> E exp[r(T-t)]

=> qS_t >E[1- exp[-r(T-t)]]

(a) qS_t =1% of Rs100=Rs 1

$E[1- \exp[-r(T-t)]]$ =Rs 150 *(1-0.9958)=Rs 0.62

Hence, the option should not be exercised early
(Total 4 marks)

(b) qS_t =1% of Rs100=Rs 1

$E[1- \exp[-r(T-t)]]$ =Rs 140 *(1-0.9876)=Rs 1.74

Hence early exercise may be advantageous.

(ii)(a) The higher the risk-free rate of interest, the better it is to exercise option early, as the interest lost by delaying the exercise is greater. Hence S_c should increase with the risk-free rate of interest.

(b) If the exercise price is increased , the immediately exercised value , (E-S) , will be greater . Hence S_c will increase with E.

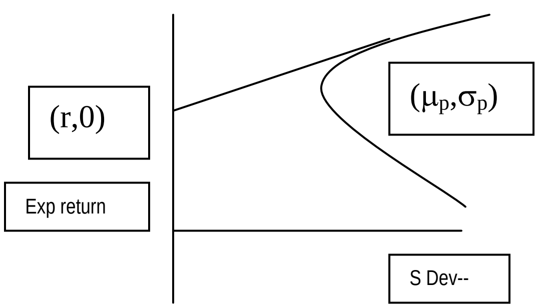
(c) If the option expires out of the money (with $S_T >> E$) it would have been better to kept hold of the share rather than exercising the option .If the share is more volatile, the probability of $S_T >> E$ is greater, so the value of the option to wait is great. Hence S_c needs to be lower for early exercise.

(Total 3 marks)

Q4 (9 marks)

SOLUTION QUESTION 4

(i)Introducing a risk-free asset that can be bought/sold in unlimited quantities makes the efficient frontier a straight line in mean-standard deviation space.



Assume optimal portfolio is x_e in equities, x_b in bonds.
σ_p = sdev portfolio, μ_p =mean return on portfolio
σ_b = sdev bond, μ_b =mean return on bond
σ_e = sdev equity, μ_e =mean return on equity

r=risk-free rate

$\mu_p = x_e\mu_e + x_b\mu_b + (1- x_e - x_b)r$ ------(1)

$\sigma_p^{\,2} = x_e^{\,2}\sigma_e^{\,2} + x_b^{\,2}\sigma_b^{\,2} + 2\, x_e\, x_b\sigma_{eb}$ ------(2)

σ_{eb} =cov(equity,bond)

Using method of Lagrange Multipliers, Lagrangian

$W = x_e^{\,2}\sigma_e^{\,2} + x_b^{\,2}\sigma_b^{\,2} + 2\, x_e\, x_b\sigma_{eb} - \lambda(x_e\mu_e + x_b\mu_b + (1- x_e - x_b)r - \mu_p)$--------(3)

Now,

$$\frac{\partial W}{\partial x_e} = 2\, x_e\sigma_e^{\,2} + 2\, x_b\sigma_{eb} - \lambda(\mu_e - r) = 0$$

$\Rightarrow 2\, x_e\sigma_e^{\,2} + 2\, x_b\sigma_{eb} = \lambda(\mu_e - r)$ ----(4)

Similarly,

$$\frac{\partial W}{\partial x_b} = 0$$

$\Rightarrow 2\, x_b\sigma_b^{\,2} + 2\, x_e\sigma_{eb} = \lambda(\mu_b - r)$ ----(5)

$$\frac{\partial W}{\partial \lambda} = 0$$

Dividing (4) by (5),

$$\frac{x_e\sigma_e^{\,2} + x_b\sigma_{eb}}{x_b\sigma_b^{\,2} + x_e\sigma_{eb}} = \frac{\mu_e - r}{\mu_b - r} = k(cons\tan t)$$

Dividing throughout by x_b and rearranging,

$$\frac{x_e}{x_b} = \frac{k\sigma_b^{\,2} - \sigma_{eb}}{\sigma_e^{\,2} - k\sigma_{eb}} ----(6)$$

$\sigma_b^2 = 0.1$^2, σ_e^2=0.2^2, σ_{eb} =0.6*0.1*0.2=0.012

k=(0.1-0.05)/(0.07-0.05)=5/2=2.5

Therefore, $\frac{x_e}{x_b} = \frac{2.5 \times 0.1^2 - 0.012}{0.2^2 - 2.5 \times 0.012} = \frac{0.013}{0.01} = 13/10$

$\therefore$ Optimal sub-portfolio of risky assets =13/23 in equities, 10/23 in bonds

(Total 7 marks)

(ii)Gradient of transformation line=(μ_p -r)/ σ_p

μ_p =13/23*0.10 + 10/23* 0.07=0.08696

σ_p =[(13/23)^2*0.2^2 +(10/23)^2*0.1^2 +2*13/23*10/23*0.012]^0.5=0.1434

Gradient of transformation line=(0.08696-0.05)/0.1434≈0.258

Breakup of marks 6 marks for 6 eqns+ 3 marks as shown above

Q5 (7 marks)

Q5
(i)

Good state price (SP_g) pays 1 in good state and 0 in bad state
So construct portfolio of *a* units of X and *b* units of Y such that:

$1.65a + 2.2b = 1$
$1.2a + 1.1b = 0$

Therefore, *a*=-1.3333 and *b*=1.4545, and SP_g is

$SP_g = aX + bY = -1.3333*(1.35) + 1.4545*(1.5) = 0.382$

Similarly, bad state price (SP_b) pays 0 in good state and 1 in bad state
So construct portfolio of *a* units of X and *b* units of Y such that:

$1.65a + 2.2b = 0$
$1.2a + 1.1b = 1$

Therefore, *a*=2.6667 and *b*=-2, and SP_b
is

$SP_g = aX + bY = 2.6667*(1.35) -2*(1.5) = 0.60$

(ii)
Expected return on X is

42%*(1.65/1.35) + 58%*(1.2/1.35) - 1 = 2.89%

Expected return on Y is

42%*(2.2/1.5) + 58%*(1.1/1.5) - 1 = 4.13%

Q6 (8 marks)

SOLUTION Q6

(i) $c + Ee^{-r(T-t)} = p + S$

(put-call parity for non-dividend paying stock)

$$\Rightarrow \frac{\partial c}{\partial t} + rEe^{-r(T-t)} = \frac{\partial p}{\partial t}$$

$$\Rightarrow \theta_c + rEe^{-r(T-t)} = \theta p$$

$$\Rightarrow \theta_p = \frac{-\sigma S \exp(-\frac{1}{2}z^2)}{2\sqrt{2\pi\tau}} + Ere^{-r(T-t)}\left[1 - \Phi(y)\right]$$

$$\Rightarrow \theta_p = Ere^{-r(T-t)}\Phi(-y) - \frac{\sigma S \exp(-\frac{1}{2}z^2)}{2\sqrt{2\pi\tau}}$$

(ii) Call options:

In the money limiting value is when S→∝, $\theta_c \to - rEe^{-r(T-t)}$

Out of the money limiting value is when S→0, $\theta_c \to 0$

Put options:

In the money limiting value is when S→0, $\theta_p \to rEe^{-r(T-t)}$

Out of the money limiting value is when S→∝, $\theta_p \to 0$

(Total 2 marks)

(iii) $\theta_p = \frac{\partial p}{\partial t}$

Hence if S is constant, we can write
$\delta p \approx \theta_p * \delta t$
for a small change in time δt

For this particular put option,
Z=[ln(0.8) +(0.05+0.5*0.2^2)]/0.2=-0.7657

$Y = z - \sigma\sqrt{\tau} = -0.9657$
Therefore θ_p =Rs 125*0.05exp(-0.05)Φ(0.9657)-(0.2*Rs100exp(-0.5*0.7657^2))/(2√π)
=Rs 1.98

Therefore δp=Rs1.98*(1/365)=Rs0.005(Ans)

Q7 (12 marks)

SOLUTION—Q7 Arbitrage Pricing Theory(12 marks)

(i) (a)

$$R_a = \mu_a + \sum_{i=1}^{N} a_i F_i + \varepsilon_a$$

R_a =return on asset a
a_i = sensitivity of asset a to index F_i
μ_a =expected return on asset a
ε_a=error term for asset a(i.e random part of return on asset a)

(i)(b)

Risk premium on asset a = $\sum_{i=1}^{N} a_i \lambda_i$

Where λ_i=risk premium on the portfolio which follows the movements in F_i

Statistical requirements are:

$E[\varepsilon_a]=0$ for all assets
$E[\varepsilon_a, \varepsilon_b]=0$ for any pair of assets
$E[\varepsilon_a, F_i]=0$ for all the factors

(Total 2 marks)

(ii) $\lambda_i = \mu_i - r$ =risk premium on portfolio I
r= risk free rate

Let x_i=allocation rate of portfolio i in market portfolio
If $a_i = \beta_a x_i, \beta_a$ =beta factor of asset a
APT formula for risk premium on asset a becomes

$$= \sum_{i=1}^{N} \beta_a x_i (\mu_i - r) = \beta_a \sum_{i=1}^{N} x_i (\mu_i - r)$$

$$= \beta_a \sum_{i=1}^{N} x_i \mu_i - \beta_a r \sum_{i=1}^{N} x_i$$

$$= \beta_a (\mu_M - r)$$

as for CAPM

μ_M=expected return on market portfolio

(iii)

(a)Eliminating F_2 gives,

R_A- R_B=0.06+2 F_1

Or, 0.5 R_A-0.5 R_B=0.03 + F_1
Or, 0.5 R_A-0.5 R_B + r=0.08 + F_1,r=risk-free rate=0.05

Therefore, portfolio which follows F_1 is $\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$ in asset A, asset B and risk-free asset
Mean return on portfolio is 0.08

Eliminating F_1 gives,

R_A- 3R_B=-0.08-4 F_2

Or, 0.75 R_B-0.25 R_A=0.02 + F_2
Or, 0.75 R_B-0.25 R_A +0.5 r=0.045 + F_2,r=risk-free rate=0.05

Therefore, portfolio which follows F_2 is $\left(\frac{-1}{4}, \frac{3}{4}, \frac{1}{2}\right)$ in asset A, asset B and risk-free asset **(**
Mean return on portfolio is 0.045
(Total 2 marks)

(b) (Repetition of mean returns derived above)
PROVED AS PART OF SOLUTION (iii)(a) ABOVE,
Mean return on portfolio following F_1 is 0.08
Mean return on portfolio following F_2 is 0.045

(c) Under the APT, λ_1=0.08-0.05=0.03, λ_2=0.045-0.05=-0.005

Risk premium on asset A=3*0.03+2*(-.005)=0.08 -----

Risk premium on asset B=0.03+2*(-.005)=0.02 -----

Q8 (5 marks)

Macroeconomic factor models
use observable economic time series as factors, e.g.:
annual inflation rates
economic growth
short term interest rates
long term governemnt bond yields
yield margins (corporates over gov bonds)
macroeconomic variables that are assumed to influence returns & security prices in practice
Fundamental factor models
similar to macroeconomic models,
but instead of (or in addition to) macroeconomic variables, they use company-specific variables e.g.:
level of gearing
price earnings ratio
level of R&D spending
industry group to which company belongs
commercially available fundamental factor models typically use many tens of factors
Statistical factor models
Don't rely on specifying factors independently of historical returns data
Factors (which ones and how many are used) are determined by the data
Principal components may be used to determine set of indices that explain as much as possible of the observed variance
However, indices may have little or no meaningful economic interpretation, and may vary considerably between different data sets

Q9 (15 marks)

9(i) No frictions; short-selling permitted; small investor (i.e. does not move the market); market is arbitrage-free; stock price is given by

$dS_t = {}_tS_t\, dt + S_t\, dZ_t$

where Z is a standard Brownian motion.

All are, in some sense, implausible. Friction (spreads and commission) is present; short-selling is available but on very different terms; small investor not true for an investment bank; stock-market returns are not compatible with normality (fat tails, jumps); arbitrages occur (for short periods).

9 (ii) a

Let N be number of options written
Then N.phi(d1)= 200000
Valus of bank's portfolio is: 120.N.phi(d1)-140.exp(-2.5%).N.phi(d2) = 120.200000-19000000 = 5000000

9(ii)b

phi(d2)/phi(d1) = 19000000/(200000.140.exp(-2.5%)) = 0.6957

If sigma=40%:	d1	(0.3152)
	d2	(0.5980)
	phi(d1)	0.3763
	phi(d2)	0.2749
	phi(d2)/phi(d1)	**0.7305**

If sigma=60%:	d1	(0.0923)
	d2	(0.5165)
	phi(d1)	0.4632
	phi(d2)	0.3027
	phi(d2)/phi(d1)	**0.6535**

Linear interpolation gives an estimate of 49.03% for the implied volatility

(5 marks)

9(ii)c

Therefore, by using this implied volatility,
N = 2,00,000 / phi(d1) = 474964 contracts

Q10 **(10 marks)**

10(ii) Delta: the rate of change in derivative price with respect to change in the price of underlying asset.

Please see below for 10(i), 10(iii)

(i)

q = 0.5418

(1 - q) = 0.4582

t=0	t=1	t=2, before dividend	t=2	t=3	Put option value	Probability		Expected value
				130.9	-	q.q.q =	0.1591	-
		121	119					
				107.1	2.90	q.q.(1-q) =	0.1345	0.39
	110			106.7	3.30	2.q.q.(1-q)	0.2690	0.89
100		99	97					
	90			87.3	22.70	2.q.(1-q).(1-q) =	0.2275	5.16
				86.9	23.10	q.(1-q).(1-q) =	0.1137	2.63
		81	79					
				71.1	38.90	(1-q).(1-q).(1-q) =	0.0962	3.74
			[1]	[1]	[1]		[1]	**12.81**

q = {exp(-0.1*3/12) - 0.9} / {1.1 -0.9 } = 0.5418 [1] 12.49

Value of put option = Rs 12.81*exp(-0.1 * 3/12) = **Rs 12.49** [1]

(iii)

Value of option at (2,1) is

2.90 * (1-q) + 0 = 1.33 1.33

Value of option at (2,2) is

3.30 * q + 22.70 * (1-q) = 12.19 12.19

Value of option at (2,3) is

23.10 * q + 38.90 * (1-q) = 30.34 30.34 [1]

$Delta_1$ = (1.33 - 12.19) / (119 - 97) = -0.49 (0.49) [1]

This applies at {s,t} = { (119+97) / 2 , 2 } = { 108, 2 }

$Delta_2$ = (12.19 - 30.34) / (97 - 79) = -1.01 (1.01) [1]

This applies at {s,t} = { (97+ 79) / 2 , 2 } = { 88, 2 }

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