Institute of Actuaries of India

Subject CT6-Stastatical Models

October/Nov 2007 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Question 1

(i) $x_1, \ldots, x_n$ are the observed claims:

$$f(\theta) \propto e^{-\frac{(\theta-\mu)^2}{2\sigma_2^2}} \propto e^{-\frac{1}{2\sigma_2^2}(\theta^2-2\theta\mu)}$$

$$p(\underline{x}|\theta) \propto \prod_{i=1}^{x} e^{-\frac{(x_1-\theta)^2}{2\sigma_1^2}}$$

$$= e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$\propto e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(\theta^2-2\theta x_i)}$$

$$= e^{-\frac{1}{2\sigma_1^2}\left(n\theta^2-2\theta\sum_{i=1}^{n}x_i\right)}$$

$$p(\theta|\underline{x}) \propto p(\underline{x}|\theta)\, p(\theta)$$

$$\propto e^{-\frac{1}{2\sigma_2^2}(\theta^2-2\theta\mu)-\frac{1}{2\sigma_1^2}\left(n\theta^2-2\theta\sum_{i=1}^{n}x_i\right)}$$

$$= e^{-\left\{\left(\frac{1}{2\sigma_2^2}+\frac{n}{2\sigma_1^2}\right)\theta^2-\left(\frac{\mu}{2\sigma_2^2}+\frac{\sum_{i=1}^{n}x_i}{2\sigma_1^2}\right)2\theta\right\}}$$

$$= e^{-\left(\frac{\sigma_1^2+n\sigma_2^2}{2\sigma_1^2\sigma_2^2}\right)\theta^2-2\theta\left(\frac{\mu\sigma_1^2+\left(\sum_{i=1}^n x_i\right)\sigma_2^2}{2\sigma_1^2\sigma_2^2}\right)}$$

$$\propto e^{-\frac{\sigma_1^2+n\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\theta-\left(\frac{\mu\sigma_1^2}{\sigma_1^2+n\sigma_2^2}+\frac{\sigma_2^2\sum_{i=1}^n x_i}{\sigma_1^2+n\sigma_2^2}\right)\right)^2}$$

$$\Rightarrow \theta|\underline{x} \sim N\left(\frac{\mu\sigma_1^2+\left(\sum_{i=1}^n x_i\right)\sigma_2^2}{\sigma_1^2+n\sigma_2^2}, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+n\sigma_2^2}\right)$$

(ii) **For the given values of the parameters,**
Mean of the posterior distribution = 42/13,
Variance of the posterior distribution = 4/13.

Hence,

$P[\theta > \mu \mid x_1,x_2,x_3] = P[(\theta - 42/13)/(4/13)^{1/2} > (2 - 42/13)/(4/13)^{1/2} \mid x_1,x_2,x_3]$
$= P[(\theta - 42/13)/(4/13)^{1/2} > -2.2188 \mid x_1,x_2,x_3]$
$= 0.987$

[Total 8]

Question 2

(i) **0% level**

P (0 claims) = $e^{-0.29}$ **= 0.7483**
P (≥1 claim) = 1 −0.7483 = 0.2517

25% level

P (0 claims) = $e^{-0.22}$ **= 0.8025**
P (≥ 1 claim) = 1 − 0.8025 = 0.1975

50% level

P (0 claims) = $e^{-0.18}$ **= 0.8353**
P (1 claim) = $0.18e^{-0.18}$ **= 0.1503**
P (≥ 2 claims) = 1 − 0.8353 − 0.1503 = 0.0144

75% level

P (0 claims) = $e^{-0.10}$ **= 0.9048**
P (1 claim) = $0.10e^{-0.10}$ **= 0.0905**
P (≥ 2 claims) = 1 − 0.9048 −0.0905 = 0.0047

Therefore the transition matrix is:

0.2517	**0.7483**	**0**	**0**
0.1975	**0**	**0.8025**	**0**
0.0144	**0.1503**	**0**	**0.8353**
0.0047	**0**	**0.0905**	**0.9048**

(ii) **In the steady state**

$$\Pi \begin{bmatrix} 0.2517 & 0.7483 & 0 & 0 \\ 0.1975 & 0 & 0.8025 & 0 \\ 0.0144 & 0.1503 & 0 & 0.8353 \\ 0.0047 & 0 & 0.0905 & 0.9048 \end{bmatrix} = \Pi$$

$0.2517\,\pi_0 + 0.1975\,\pi_1 + 0.0144\,\pi_2 + 0.0047\,\pi_3 = \pi_0$

$0.7483\,\pi_0 + 0.1503\,\pi_2 = \pi_1$

$0.8025\,\pi_1 + 0.0905\,\pi_3 = \pi_2$

$0.8353\,\pi_2 + 0.9048\,\pi_3 = \pi_3$

and $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1.$

Hence

$\pi_3 = 8.7742\,\pi_2$

$\pi_2 = 3.8969\,\pi_1$

$\pi_1 = 1.8062\,\pi_0$

$\pi_0 = 0.0139$

$\pi_1 = 0.0252$

$\pi_2 = 0.0983$

$\pi_3 = 0.8626$

[Total 10]

Question 3

(i) The probability distribution for Z is:

$$P(Z = z) = \binom{n}{z} \mu^z (1-\mu)^{n-z}, \quad z = 0,1,2,\ldots,n$$

We now want the probability distribution for $Y = Z/n$. Making the substitution $z = ny$, we get:

$$P(Y = y) = \binom{n}{ny} \mu^{ny} (1-\mu)^{n-ny}, \qquad y = 0,\ 1/n,\ 2/n,\ \ldots,\ 1$$ [½]

We now rewrite this in the form of an exponential family:

$$\binom{n}{ny}\mu^{ny}(1-\mu)^{n-ny} = \exp\left[n\left\{y\log\mu + (1-y)\log(1-\mu)\right\} + \log\binom{n}{ny}\right]$$

$$= \exp\left[n\left\{y\log\frac{\mu}{1-\mu} + \log(1-\mu)\right\} + \log\binom{n}{ny}\right] \quad [½]$$

This is now in the form of an exponential family:

$$f(y) = \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right]$$

where $\theta = \log\left(\frac{\mu}{1-\mu}\right)$, $\phi = n$, $a(\phi) = 1/\phi$, $b(\theta) = \log(1+e^{\theta})$ and $c(y,\phi) = \log\binom{n}{ny}$. [1]

(ii) The natural parameter is a function of the mean response μ. It is the function $\theta(\mu)$ calculated in part (i):

$$\theta = \log\left(\frac{\mu}{1-\mu}\right) \quad [1]$$

and the canonical link function is $g(\mu) = \theta(\mu) = \log\left(\frac{\mu}{1-\mu}\right)$. [1]

The variance function is given by differentiating the function $b(\theta)$ twice with respect to θ. We have:

$$b'(\theta) = \frac{e^{\theta}}{1+e^{\theta}}$$

and: $$b''(\theta) = \frac{(1+e^{\theta})e^{\theta} - e^{2\theta}}{(1+e^{\theta})^2} = \frac{e^{\theta}}{(1+e^{\theta})^2}$$

We now write this back in terms of μ:

$$\text{var}(\mu) = b''(\theta) = \frac{e^{\theta}}{\left(1+e^{\theta}\right)^2} = \mu(1-\mu) \quad [2]$$

The variance is $b''(\theta) / a(\phi) = \mu(1-\mu)/n$.

Question 4

(i) **The assumptions underlying the basic chain-ladder method are:**

- **Payments in the earliest accident year are fully developed.**
- **Payment from each accident year will develop in the same way.**
- **Weighted average past inflation will be repeated in future.**

(ii) **Step 1: Adjust the incremental data to constant money terms**

		settlement delay in years (development year)				
		0	1	2	3	4
year of accident (origin year)	2004	157.8	123.9	73.0	33.9	9.0
	2005	142.9	110.1	60.4	27.0	
	2006	166.3	131.4	75.0		
	2007	151.6	110.0			
	2008	138				

Step 2: form cumulative data, i.e. the sum to date along the rows

		settlement delay in years (development year)				
		0	1	2	3	4
year of accident (origin year)	2004	157.8	281.7	354.7	388.6	397.6
	2005	142.9	253.0	313.5	340.5	
	2006	166.3	297.7	372.7		
	2007	151.6	261.6			
	2008	138.0				

Step 3: Calculate the chain ladder ratios

r_{01}	r_{12}	r_{23}	r_{34}
1.7685	1.2504	1.0912	1.0232

Step 4: apply chain ladder ratios to project future cumulative data

		settlement delay in years (development year)				
		0	1	2	3	4
year of accident (origin year)	2004					
	2005					348.3
	2006				406.7	416.1
	2007			327.1	356.9	365.2
	2008		244.1	305.2	333.0	340.7

Step 5: derive incremental data from cumulative data

		settlement delay in years (development year)				
		0	1	2	3	4
year of accident (origin year)	2004					
	2005					7.9
	2006				34.0	9.4
	2007			65.5	29.8	8.3
	2008		106.1	61.1	27.8	7.7

Step 6: project to allow for future inflation *(Inflation is 2% for the first year and 4% thereafter).*

		settlement delay in years (development year)				
		0	1	2	3	4
year of accident (origin year)	2004					
	2005					8.0
	2006				34.7	10.0
	2007			66.8	31.6	9.1
	2008		108.2	64.8	30.7	8.7

Step 7: add up the numbers to give the estimate you require which comes out to be Rs. 372,600.

[Total 11]

Question 5

(i)

The prior distribution for θ has the form:

$$\text{Prior}(\theta) \propto \theta^{\beta-1}(1-\theta)^{\beta-1}$$

The random variable X has a binomial distribution with parameters m and θ. So the likelihood function based on a random sample $x_1, \ldots, x_n$ is:

$$L(\theta) = \binom{m}{x_1}\theta^{x_1}(1-\theta)^{m-x_1} \times \binom{m}{x_2}\theta^{x_2}(1-\theta)^{m-x_2} \times \cdots \times \binom{m}{x_n}\theta^{x_n}(1-\theta)^{m-x_n}$$
$$\propto \theta^{\sum x_i}(1-\theta)^{mn-\sum x_i} \qquad [1]$$

The posterior distribution for θ is proportional to the product of the likelihood function and the prior distribution:

$$\text{Posterior}(\theta) \propto \theta^{\beta-1}(1-\theta)^{\beta-1} \times \theta^{\sum x_i}(1-\theta)^{mn-\sum x_i}$$
$$= \theta^{\beta-1+\sum x_i}(1-\theta)^{\beta-1+mn-\sum x_i}$$

We see that this has the form of another beta distribution, this time with parameters $\beta + \sum x_i$ and $\beta + mn - \sum x_i$. [2]

(ii)

The Bayesian estimate under quadratic loss is the mean of the posterior distribution. We know that the posterior distribution is another beta distribution with parameters $\beta+\sum x_i$ and $\beta+mn-\sum x_i$. So the Bayesian estimate for θ is (using the formula for the mean of the beta distribution given in the *Tables*):

$$\hat{\theta}=\frac{\beta+\sum x_i}{\beta+\sum x_i+\beta+mn-\sum x_i}=\frac{\beta+\sum x_i}{2\beta+mn} \qquad [1]$$

We can split this up into a credibility estimate as follows:

$$\begin{aligned}\frac{\beta+\sum x_i}{2\beta+mn}&=\frac{\beta}{2\beta+mn}+\frac{\sum x_i}{2\beta+mn}\\&=\frac{\beta}{2\beta}\times\frac{2\beta}{2\beta+mn}+\frac{\sum x_i}{mn}\times\frac{mn}{2\beta+mn}\\&=Z\times\frac{\sum x_i}{mn}+(1-Z)\times\mu\end{aligned} \qquad [2]$$

where $Z=\dfrac{mn}{2\beta+mn}$ and $\mu=½$. The value of ½ is the mean of our prior distribution for θ. Z is the credibility factor. [1]

It is easy to see that the quantity $\sum x_i/(mn)$ maximizes the log of the likelihood $L(\theta)$ given in part (i), as the derivative is zero when θ takes this value, and the second derivative is negative.

(iii) **When β = 5, the prior variance is (using the formula for the variance of the beta distribution given in the Tables):**

=(5*5) /[((5+5)^2)*(5+5+1)]

= 0.0227

When β =15, the prior variance is (using the formula for the variance of the beta distribution given in the Tables):

=(15*15)/[((15+15)^2)*(15+15+1)]

= 0.0081

So when β is larger we have a smaller prior variance. This corresponds to the situation where we are more certain about the value of θ.

Both prior distributions have a mean of ½, so we think that the value of θ might be somewhere near ½. However, if we choose a larger value β. we are saying that we are more certain that the value of θ .is close to ½.
This means that we want to put more emphasis on the prior distribution in our analysis, and less emphasis on the data available from the sample. This means that we need a smaller value for Z , and this is in fact the case. Z is smaller when $\beta = 15$ than when $\beta = 5$.

Question 6

(i) **A stochastic process is weakly stationary if it has constant mean and the covariance is constant for each fixed lag.**

(ii) **The moving average process is $X_n = Z_n + \beta Z_{n-1}$.**

The mean of the process is $E[X_n] = (1+\beta)\,\mu$. This is constant.

The variance:

$$\text{Var}(X_n) = \text{Var}(Z_n + \beta Z_{n-1}) = (1+\beta^2)\,\sigma^2.$$

The covariance

$$\text{Cov}(X_n, X_{n+1}) = \text{Cov}(Z_n + \beta Z_{n-1}, Z_{n+1} + \beta Z_n) = \beta\sigma^2.$$

For $m > 1$, $$\text{Cov}(X_n, X_{n+m}) = \text{Cov}(Z_n + \beta Z_{n-1}, Z_{n+m} + \beta Z_{n+m-1}) = 0.$$

The covariance at higher lags are 0 since there is no overlap between the Z's. The covariance at negative lags are the same as those at the corresponding positive lags. Since none of these expressions depends on n, it follows that the process is weakly stationary.

(iii) **Write the model equation as**

$(1 - 1.5B + .5B^2)X = Z,$

where B is the back-shift operator. The polynomial in B factorizes as

$(1 - B)(1 - .5B),$

Since one of the roots of the polynomial has magnitude 1, the process is NOT stationary.

(iv) **The process X is ARIMA(1,1,0), so $(1-B)X$ is AR(1). Define the process Y as $Y = (1-B)X$, and write this AR(1) process as**

$(1 - .5B)Y = Z$.

According to standard formulae, the autocorrelation at lag 1 is 0.5.

[Total 9]

Question 7

(i) **The stored table of random numbers generated by a physical process may be too short, a combination of linear congruential generators (LCG) can produce a sequence which is infinite for practical purposes.**
It might not be possible to reproduce exactly the same series of random numbers again with a truly random number generator unless these are stored.

A LCG will generate the same sequence of numbers with the same seed.

Truly random numbers would require either a lengthy table or hardware enhancement compared with a single routine for pseudo random numbers.

(ii) **The requirement to calculate cos and sin in the Box-Muller method is time consuming for a computer. The Polar Method avoids this by using the acceptance-rejection method.**

(iii)

(a) **If this is not available as a built in "RAND" function, we can use a linear congruential generator (based on published parameter values).**

(b) **The chi square distribution with 6 degrees of freedom is the same as a gamma distribution with parameters α = 3 and λ = 0.5 . Using the additive property of the exponential distribution, we could generate sets of 3 values from the exponential distribution with mean 2, using the method in (b), and add these together to get a single value from χ^2_6 (Chi square with 6 degree of freedom).**

Alternatively, we could generate 6 independent N(0,1) values, square them and add them up. This uses the fact that the square of a standard normal variable is χ^2_1 (Chi square with 1 degree of freedom), and the chi square distribution has an additive property.

(c) **We can generate samples from an exponential with mean 5, using the inverse distribution function method, i.e., calculate**

$X = -0.2 \log(1-u)$ or $X = -0.2 \log u$,
[1]

u being a sample from the uniform distribution. Then, we can use the "memoryless" property of the exponential distribution, which tells us that X–10 | X>10 has the same distribution as the original X i.e., exponential with mean 5. Thus, we only have to add 10 to each sample to obtain samples from X | X>10.

(d)

The distribution function for the Burr distribution is $F(x)=1-\lambda^{\alpha}(\lambda+x^{\gamma})^{-\alpha}$. So we can apply the inverse distribution function method by calculating $x=\left[\lambda(u^{-1/\alpha}-1)\right]^{1/\gamma}$. [1]

[Total 8]

Question 8

(i) **In this case, taxi driver A has to take a decision in the face of uncertainty. His choices consist of selecting customer C or D. The nature's choice consist of the arrival of customer C before that of customer D, or the other way 'round. The 'statistician' (driver A) learns about the probabilities of nature's choices from the delay distributions, which are presumably based on past data.**

(ii) **Let $\theta = 1$ correspond to the event that C arrives before D and $\theta = 2$ correspond to the event that D arrives before C.**

Let d_1 be the decision of serving the customer arriving first, and d_2 be the decision of serving the customer arriving second.

Then the loss matrix for driver A is as follows.

	$\theta = 1$	$\theta = 2$
d_1	–50	–100
d_2	–100	–50

(iii) **From the joint distribution of X and Y (which is simply the product of their respective marginal distributions), the probability of customer C arriving first is**

$$\begin{aligned}
P(\theta = 1) &= P(X < Y + 15) \\
&= \int_0^\infty P(X < Y + 15 \,|\, Y = y) f_Y(y) dy \\
&= \int_0^\infty P(X < y + 15) f_Y(y) dy \\
&= \int_0^\infty [1 - e^{-(y+15)/30}] \frac{1}{15} e^{-y/15} dy \\
&= 1 - \frac{e^{-1/2}}{15} \int_0^\infty e^{-y/10} dy \\
&= 1 - \frac{2e^{-1/2}}{3} \\
&= 0.5956.
\end{aligned}$$

The probability of customer D arriving first is 1 – 0.5956 = 0.4044.

(iv)

Expected loss for decision d_1 is

$$\begin{aligned}
&-50P(\theta = 1) - 100P(\theta = 2) \\
&\quad = -50 * 0.5956 - 100 * 0.4044 \\
&\quad = -70.22.
\end{aligned}$$

Likewise, the expected loss for decision d_2 is

$$\begin{aligned}
&-100P(\theta = 1) - 50P(\theta = 2) \\
&\quad = -100 * 0.5956 - 50 * 0.4044 \\
&\quad = -79.78.
\end{aligned}$$

The Bayes' decision rule for driver A is to wait for the second customer.

[Total 9]

Question 9

(i) **Expected value of loss = 0.06 * 1 crore = Rs. 6 lakhs.**

For 100% profit, premium should be Rs. 12 lakhs.

(ii) **Insurer's surplus at the end of the t^{th} launch is**

$$U(t) = 0.5 + ct - S_t,$$

Where S_t is the number of claims/unsuccessful placements.

If $c = 0.12$, then $U(1) = 0.5 + 0.12 - S_1$. This expression can be negative if and only if $S_1 = 1$. The probability of this event is 0.06.

(iii) **The number of launches required for the first failure (N) has the geometric distribution with parameter $p = 0.06$.**

Therefore, the probability of ruin at the first claim is

$$\begin{aligned}
P[U(N) < 0] &= P[0.5 + cN - S_N < 0] \\
&= P[0.5 + 0.12N - 1 < 0] \\
&= P[N < 0.5/0.12] \\
&= P[N = 1] + P[N = 2] + P[N = 3] + P[N = 4] \\
&= p + (1 - p)p + (1 - p)^2 p + (1 - p)^3 p \\
&= p * [1 - (1 - p)^4] / [1 - (1 - p)] \\
&= 1 - (1 - p)^4 \\
&= 1 - (1 - 0.06)^4 \\
&= 0.219.
\end{aligned}$$

[Total 7]

Question 10

- **The premium rating process for the general insurance business starts with the calculation of the risk premium.**
- **To calculate the office premium, the risk premium is loaded for commission, expenses, other contingencies and the profit.**
- **Alternatively, where the established rating structure exists,**
  - **the process is to identify the changes that need to be made in the relative levels of premium for different categories within that structure, and then**
  - **to determine the overall percentage adjustment that needs to be applied to the existing premiums to achieve the desired financial result.**

[Total 4]

Question 11

(i) **The total claim amount is $S = X_1 + X_2 + \ldots + X_N$, where N is the number of claims and $X_1, X_2, \ldots, X_N$ are successive claim sizes. The MGF of S is**

$$\begin{aligned} M_S(t) &= E\left(e^{(X_1+\cdots+X_N)t}\right) \\ &= E_N\left[E\left(e^{(X_1+\cdots+X_N)t}|N\right)\right] \\ &= E_N\left[E\left(e^{X_1 t}\right)\right]^N \\ &= E_N\left[M_X(t)\right]^N \\ &= \sum_{n=0}^{\infty}\left[M_X(t)\right]^n \frac{e^{-\lambda}\lambda^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{e^{-\lambda}\left[M_X(t)\lambda\right]^n}{n!} \\ &= e^{-\lambda}e^{M_X(t)\lambda} \\ &= \exp\left[\lambda\left(M_X(t)-1\right)\right]. \end{aligned}$$

(ii) **The mean of the aggregate annual claims is**

$$\begin{aligned} M_S'(t)\,|_{t=0} &= \exp\left[\lambda\left(M_X(t)-1\right)\right]\lambda M_X'(t)\big|_{t=0} \\ &= \lambda M_X'(0) \\ &= 10 * 40 \\ &= 400. \end{aligned}$$

(iii) **The second moment of the aggregate annual claims is**

$$\begin{aligned} M_S''(t)\,|_{t=0} &= \exp\left[\lambda\left(M_X(t)-1\right)\right]\left[\lambda M_X''(t) + \lambda^2\left(M_X'(t)\right)^2\right]\Big|_{t=0} \\ &= \lambda M_X''(0) + \lambda^2\left(M_X'(0)\right)^2 \\ &= 10 * 2500 + 10^2 * 40^2. \end{aligned}$$

Hence, the variance is

$$E\left(S^2\right) - \left[E(S)\right]^2 = (25,000 + 10^2 * 40^2) - 400^2 = 25,000.$$

(iv) **Third moment of the aggregate annual claims is**

$$\begin{aligned} & M_S'''(t)\,|_{t=0} \\ &= \exp\left[\lambda\left(M_X(t)-1\right)\right]\left[\lambda^3\left(M_X'(t)\right)^3 + 3\lambda^2 M_X''(t)M_X'(t) + \lambda M_X'''(t)\right]\Big|_{t=0} \\ &= \lambda^3\left(M_X'(0)\right)^3 + 3\lambda^2 M_X''(0)M_X'(0) + \lambda M_X'''(0) \\ &= 10^3 * 40^3 + 3 * 10^2 * 2500 * 40 + 10 * 80,000 \\ &= 94,800,000. \end{aligned}$$

[Total 9]

Question 12

(i)

- **Under a proportional reinsurance arrangement,**
    - **The direct writer (i.e. the original insurance company) and the reinsurer share the cost of all claims for each reinsured risk.**
    - **The direct writer pays a premium to effect this reinsurance. This premium may, or may not, be calculated on the same basis as the premiums charged by the insurer.**
- **Proportional reinsurance operates in two forms:**
    - **With *quota share* reinsurance, the proportions are the same for all risks.**
    - **With *surplus* reinsurance, the proportions can vary from one risk to the next.**

(ii)

$$\text{Variance} = \lambda \times E[\text{net claim}^2]$$

$$= \lambda \times 0.8^2 \left\{ \frac{4 \times 300 \times 300}{3 \times 3 \times 2} + \left(\frac{300}{3}\right)^2 \right\}$$

$$= \lambda \times 3 \times 100^2 \times 0.8^2 = 3\lambda \times 80^2$$

$$= 288{,}000 \rightarrow \lambda = 15$$

[Total 6]

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