Institute of Actuaries of India

Subject CT4 - Models

October/Nov 2007 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Q.1)
i) A stochastic process X_n is stationary if the joint distributions of X_{t1}, X_{t2},……..X_{tm} and X_{t1+k}, X_{t2+k},……..X_{tm+k} are identical for all t1,t1,………tm, k+t1,k+t2,……k+tm $\in$ J and all integers m.

ii) The process is weakly stationary if the expectations $E[X_t]$ are constant with respect to and the covariances cov (X_t, X_{t+k}) depend only on the lag k.

iii) If t and t+u are in J then the increment for time u will be $X_{t+u} - X_t$. Often u=1 is used.

iv) For a discrete process the Markov property requires that:

$$P[X_t=x \mid X_{t1}=x_1, X_{t2}=x_2,………,X_{tm}=x_m]=P[X_t=x \mid Xt_m=x_m]$$

for all times $t_1<t_2<………<t_m<t \in J$ and all states $x_1,x_2,……x_m$, x $\in$ S.

Total - [4]

Q. 2)
i. Discrete time, discrete state space: Markov chain, simple random walk
ii. Discrete time, continuous state space: General random walk, time series
iii. Continuous time, discrete state space: Poisson process, Markov jump process
iv. Continuous time, continuous state space: Brownian motion, Ito process

Total - [4]

Q. 3)

(a) H_0: the observed transition rates $\hat{\mu}_{x+1/2}$ come from a population in which the standard table rates are the true rates.

$\aleph^2$ with m degrees of freedom

(b) H_0: the observed transition rates $\hat{\mu}_{x+1/2}$ come from a population in which the graduated transition rates are the true rates.

In the graduation process, four parameters have been estimated so sampling distribution is $\aleph^2$ with m - 4 degrees of freedom.

Total [4]

Q. 4)
i. All transition probabilities need to lie in the interval [0, 1].
Now,
$$1-2\alpha- \alpha^2 \leq 1 - \alpha- \alpha^2 \leq 1 \text{ for all } \alpha \geq 0.$$
Thus it suffices to ensure that
$$1 - 2 \alpha - \alpha^2 \geq 0 \text{ i.e. } \alpha \leq \sqrt{2} - 1.$$

So, the possible values of α is $[0, \sqrt{2} - 1]$.

ii. The chain is not irreducible since the state D is an absorbing state and it is a periodic.

Total - [4]

Q. 5)

    (i)   Female smoker aged 30 at entry.

    (ii) $\dfrac{h_j(t)}{h_i(t)} = \dfrac{\exp(-.05)}{\exp(0.1)} = 0.86070$

           Where j is male smoker aged 30 at entry and i is female smoker aged 40 at entry

           But s(t) = exp ($-\int_0^t h(s)ds$) hence

$$s_j(t) = (s_i(t))^{0.86070}$$

           which implies that

$$s_j(t) > s_i(t) \text{ for all } t > 0$$

    (iii) $\dfrac{h_j(t)}{h_i(t)} = \dfrac{\exp(0.2)}{\exp(0.05)} = 1.161$

           Where j is male smoker aged 30 at entry and i is male smoker aged 40 at entry

           But s(t) = exp ($-\int_0^t h(s)ds$) hence

$$s_j(t) = (s_i(t))^{1.161}$$

           Which implies that

$$s_j(t) < s_i(t) \text{ for all } t > 0$$

Total - [5]

Q. 6)

i.

$$P = \begin{pmatrix} \tfrac{1}{4} & \tfrac{1}{2} & \tfrac{1}{4} \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} \\ \tfrac{3}{4} & \tfrac{1}{4} & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 8/16 & 3/16 & 5/16 \\ 8/16 & 6/16 & 2/16 \\ 5/16 & 6/16 & 5/16 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 29/64 & 21/64 & 14/64 \\ 26/64 & 18/64 & 20/64 \\ 32/64 & 15/64 & 17/64 \end{pmatrix}$$

ii.
a. $P^2 23 = 2/16 = 1/8 = 0.125$

b. $(14/31) P^3 13 + (9/31) P^3 23 + (8/31) P^3 33 = (14*14+9*20+8*17)/(31*64) = 8/31$

iii. If we make n = 300, the effects of the starting point would have worn off. The answer in both the cases in the above statement would then be indistinguishable from the stationary probability π_3 in both the cases.

It can then easily be observed that the distribution in (b.) would become stationary and hence

$$\pi_3 = 8/31$$

Total - [8]

Q. 7)
(i) Using the Binomial Model of Mortality we know that

$$\theta_x \sim B(E_x, q_x)$$

And
$$Var(\theta_x) = E_x q_x (1 - q_x)$$

So
$$Var(\tilde{q}_x) = var(\theta_x / E_x) = E_x q_x (1 - q_x) / E_x^2$$

$$= q_x (1 - q_x) / E_x$$

Now if q_x is very small, which will be true for most ages then

$$Var(\tilde{q}_x) = q_x / E_x$$

Which can be estimated by

$$\tilde{q}_x / E_x = \theta_x / E_x^2$$

So the observed value of the estimator is

$$SE(\tilde{q}_x) = \sqrt{d_x} / E_x$$

(ii) Standard errors of the estimated initial rates of mortality will be large if

E_x is small relative to $\sqrt{d_x}$

For middle range ages e.g. $25 \le x < 80$, E_x will tend to be large, as these are the ages which for most investigations have a large in-force. Precise estimates will usually be produced.

At younger ages E_x will tend to be small (limited amounts of business written), so standard errors will increase, less precise estimates.

At old ages E_x will small relative to number of deaths, so standard errors will increase, less precise estimates.

The relative size of the standard error (coefficient of variation) is also important

i.e. $\sqrt{d_x} / E_x$ $d_x / E_x = 1/\sqrt{d_x}$

This will be larger for younger and older ages, where d_x is smaller

(iii) $\overset{0}{q_x} \pm 2\sqrt{d_x} / E_x$

represents an approximate 95% confidence interval for the true initial rate of mortality at age x.

If we plot these on a scatter plot of observed initial rates of mortality then the "band" formed by these standard error bars provides a guide to where the graduation curve should be drawn. For about 19 out of every 20 ages the curve drawn should lie inside the constructed "band".

Total - [8]

Q. 8)
(i) The consistency condition is $P[T_y \le t] = P[T_x \le (y-x) + t \mid T_x > (y-x)]$

(ii)

(a) $\mu_y = \lim_{dt \to 0} P[T_0 \le x+t+dt] \mid T_0 > x+t] / dt$

(b) $^*\mu_y = \lim_{dt \to 0} P[T_x \le t+dt] \mid T_x > t] / dt$

(c)

$\lim_{dt \to 0} P[x+t < T_0 \le x+t+dt] / dt$

$\mu_y = $ ---

$P[T_0 > x+t]$

$\lim_{dt \to 0} P[T_0 \le x+t+dt] / dt - P[T_0 \le x+t]$

$$\mu_y = \frac{}{P[T_0 > x+t]}$$

$$\lim_{dt \to 0} \left(P[T_0 \le x+t+dt] - P[T_0 \le x]\right) / dt - \left(P[T_0 \le x+t] - P[T_0 \le x]\right)$$

$$\mu_y = \frac{}{P[T_0 > x+t]}$$

$$\lim_{dt \to 0} \frac{\left(P[T_0 \le x+t+dt] - P[T_0 \le x]\right) / dt - \left(P[T_0 \le x+t] - P[T_0 \le x]\right)}{P[T_0 > x]} \qquad P[T_0 > x]$$

$$\mu_y = \frac{}{\dfrac{P[T_0 > x+t]}{P[T_0 > x]}}$$

$$\lim_{dt \to 0} P[T_0 \le x+t+dt \mid T_0 > x] / dt - P[T_0 \le x+t \mid T_0 > x]$$

$$\mu_y = \frac{}{P[T_0 > x+t \mid T_0 > x]}$$

$$\lim_{dt \to 0} P[T_x \le t+dt] / dt - P[T_x \le t]$$

$$\mu_y = \frac{}{P[T_x > t]}$$

$$\mu_y = \lim_{dt \to 0} P[T_x \le t+dt \mid T_x > t] / dt$$

$$= {}^* \mu_y$$

(iii) $\dfrac{\delta}{\delta s} {}_sP_x = - {}_sP_x \; \mu_{x+s}$

$\therefore \quad \dfrac{\dfrac{\delta}{\delta s} {}_sP_x}{{}_sP_x} = - \mu_{x+s}$

$\therefore \quad \dfrac{\delta}{\delta s} (\log {}_sP_x) = - \mu_{x+s}$

$$\therefore \log{}_t \mathrm{P}_x = - \int_0^t \mu_{x+s} ds + c$$

$$\therefore {}_t \mathrm{P}_x = \mathrm{e}^c . \quad exp \{ - \int_0^t \mu_{x+s} ds \}$$

From ${}_0 \mathrm{P}_x = 1$, we get c = 0 , hence the result .

Total - [9]

Q. 9)

(i) The most appropriate rate interval to use (for lives classified *x*) is the policy year rate interval starting on the policy anniversary where lives are aged *x* next birthday.

The reason is that this corresponds to the definition of the deaths and the rate is more sensitive to errors in approximation of the numerator than the denominator.

The average age at the start of the rate interval is $x - \dfrac{1}{2}$ assuming that birthdays are uniformly distributed over the policy year.

(ii)

We will use the following symbols:

$P_{x,t}$ to represent the in force at time *t* from the 1 January 1997 classified *x* next
        birthday on policy anniversary nearest to time *t*

$\theta_{x,t}$ to represent the deaths in the calendar year 1997 aged *x* next birthday on policy
      anniversary (= age next birthday at entry plus curtate duration at date of death)
      before death

$\mathrm{E}_x , \mathrm{E}_x^c$ to represent the initial and central exposed to risk respectively of lives age x
        last birthday on previous policy anniversary .

$\mathrm{P}_x (t)$ to represent the in force at time *t* from the 1 January 1997 classified *x* next
        birthday on the policy anniversary preceding time *t*.

Now $\mathrm{P}_x (t) = \dfrac{1}{2} (P_{x,t} + P_{x+1,t})$

assuming that policy anniversaries are uniformly distributed over the calendar year

$$\mathrm{E}_x^c = \int_0^{10} P_x(t) dt = \frac{1}{2} \sum_{t=0}^{9} (P_x(t) + P_x(t+1))$$

assuming that the in force population varies linearly between the dates of the investigation .

$$\mathrm{E}_x = \mathrm{E}_x^c + \frac{1}{2} \sum_{t=0}^{10} \theta_{x,t} `$$

assuming that in aggregate the deaths occur on average halfway through the policy year.

(iii)

(a) This violates the assumption that birthdays are uniformly distributed over the policy year and this would alter the age to which the rates apply. [In this case the average
age at the start of the rate interval is $x - \frac{1}{4}$]

b) This violates the assumption that the policy anniversaries occur uniformly over the calendar year, and hence the approximation

$$P_x(t) = \frac{1}{2}(P_{x,t} + P_{x+1,t})$$

, is no longer correct.
[A close approximation would now be $P_x(t) = P_{x+1,t}$ assuming an insignificant number of

policies have their anniversaries in the second half of the calendar year.]

This will also almost certainly violate the assumption that $P_x(t)$ varies linearly between the

investigation period due to the non uniform distribution of policy anniversaries over each calendar year.

Total - [11]

Q. 10)
(a) **Smoothness test**
Calculate 2nd and 3rd finite differences between the graduated rates. The more gradual the progression, and the smaller the absolute values of the third differences for a given graduation attempt, then the smoother the curve. The latter can be summarised by calculating:

$$S = \sum_x \left| \Delta^3 q^v_x \right|$$

Where $\Delta^3 q^v_x$ is the third order finite difference between the graduated rates x .

Smaller S implies greater smoothness, used for assessing the relative smoothness of each graduation attempt of a given set of crude mortality rates.

b) **Chi-squared test**
A null hypothesis (H_0) is postulated, which states that the graduated rates of mortality

q^v_x) are equal to the true underlying rates of mortality q_x at each age.

According to $H_0 : C^2 = \sum_x (\theta_x - E_x q^v_x)^2 / E_x q^v_x p^v_x \sim \aleph^2_{(n-k)}$

Where θ_x = deaths at age x

E_x = exposed to risk at age x

n = number of age groups

k = number of constraints imposed by the graduation on the distribution of C^2

Calculate the observed value of C^2 (call this c^2) and, using Chi squared tables for n-k degrees of freedom, assess:

$$\Pr(\aleph^2_{n-k} \geq c^2)$$

If this is less than, say 5%, then H_0 may be rejected. This is a test of the overall magnitude of the deviations between the crude and graduated mortality rates. If H_0 is rejected, then the implication is that over some or all of the age range the deviations are probably too large to be explained solely as due to random error. The graduation would then need to be repeated with graduated rates passing closer to the crude rates.

(c) **Serial correlation test**

According to H_0

$$R = (n/n-1) \left\{ \cfrac{\sum\limits_{i=1}^{n-1}(z_i - \bar{z})(z_{i+1} - \bar{z})}{\sum\limits_{i=1}^{n}(z_i - \bar{z})^2} \right\} \sim N(0, 1/n)$$

Where $Z_i = \cfrac{\theta_{x_i} - E_{x_i}\overset{v}{q}_{x_i}}{\sqrt{E_{x_i}\overset{v}{q}_{x_i}\overset{v}{p}_{x_i}}}$

$$\bar{z} = (1/n)\sum_{i=1}^{n} z_i$$

and x_i is the ith age group.

Calculate the observed value of R for the investigation (call this r). Then calculate:
$$\Pr\{R \geq r\}$$

H_0 would be rejected if this is too small (say less than 5%).

Procedure would be:

calculate: $z(r) = r\sqrt{n}$

H_0 would be rejected (at 5% level) if $z(r) > 1.645$.

This test is sensitive to the distribution of deviations from age to age. If there is a strong positive association between deviations from age to age (i.e. a high value of r) then the fitted curve is of the wrong shape. The graduation would be repeated using a differently shaped curve.

(ii)
(1) Plot the crude rates on graph paper.
(2) Draw as smooth a curve as possible, by hand, which appears to produce the required degree of adherence to data.
(3) Test for adherence to data and smoothness.
(4) Repeat the above by drawing different curves, making improvements to adherence and/or smoothness as indicated by the results of the tests on previous attempts.
(5) Once satisfied with the overall position and smoothness of the curve, the graduated rates can be hand polished. This means making any minor adjustments to the graduated rates, which will produce improved smoothness (e.g. in the progress of the 2nd differences) without materially altering the adherence to data.

Under (2) above, the drawing of the curve may be made easier by plotting a corridor of 95% confidence intervals, such that the resulting graduated rates would not pass outside the corridor at more than about 5% of the age range.

Also grouped rates might be calculated, producing a smoother curve but with fewer points, which can make the drawing of the curve easier.

(iii)
The graphic method is entirely flexible: it can produce any desired shape of curve. This is its fundamental advantage over the other two methods. So it would be preferred when neither of the other methods produces the desired adherence to data.

Method therefore suited to internal use (e.g. by life office or pension scheme) particularly for graduating experience for which there may be no suitable standard table: e.g. pension scheme decrements; life office withdrawal rates, etc.

Total - [13]

Q. 11)
i. X(t) is not Markov because, for example, P[Xt+1 = 2|Xt = 3,Xt−1 = 2, ...] cannot be reduced to P[Xt+1 = 2|Xt = 3].

ii. a. We need to add more states. Define the following new states
3a = Policyholder in level 3 this year given that the policyholder was in level 2 last year
3b = Policyholder in level 3 this year given that the policyholder was in level 4 last year

b. The transition matrix is then given by

	1	2	3a	4	3b
1	0.2	0.8	0	0	0
2	0.2	0	0.8	0	0
3a	0	0.2	0	0.8	0
4	0	0	0	0.8	0.2
3b	0.2	0	0	0.8	0

c. We have

$\pi_1 = 0.2\ \pi_1 + 0.2\pi_2 + 0.2\ \pi_{3b}$ $=> \pi_1 = 0.25\ \pi_2 + 0.25\ \pi_{3b}$
$\pi_2 = 0.8\ \pi_1 + 0.2\ \pi_{3a}$ $=> \pi_2 = 0.25(\pi_{3a} + \pi_{3b})$
$\pi_{3a} = 0.2\ \pi_{3b} + 0.2\ \pi_{3a}$ $=> \pi_{3a} = 0.25\ \pi_{3b}$
$\pi_{3b} = 0.2\ \pi_4$

We also have $\sum \pi_i = 1$. Thus, we have

$(\pi_1, \pi_2, \pi_{3a}, \pi_4, \pi_{3b}) = (21, 20, 16, 320, 64) /441$

Thus, the long run probability of the policyholder being in level 3 is

$(16 + 64)/441 \quad = \quad 80 / 441.$

Total - [14]

Q. 12)
Assume the usual notation. Also, more than one event occurring in time (t, t+dt) is o(dt) in
probability.

Let qij(t) denote the (i, j)[th] element of the transition matrix. We need to consider only the
elements $q_{i,i+1}(dt)$, $q_{i,i}(dt)$ and $q_{i,i-1}(dt)$ with special attention to the cases i = 0 and i = m.
Note that our chain X_t is the number of machines working at time t.
Now, for $1 \le i \le n$, we have

$q_{i,i+1}(dt) = \lambda dt + o(dt)$

$q_{i,i-1}(dt) = \begin{bmatrix} i \\ 1 \end{bmatrix} (\mu dt)\ (1 - \mu dt)^{i-1} + o(dt) = i\mu dt + o(dt)$

Note that there are i machines and 1 breakdown occurs and also expand and everything o(dt) to
the end

$=> q_{i,i}(dt) = 1 - (\lambda + i\mu)\ dt + o(dt)$

Thus, at i = 0
                    $q_{0,1}(dt) = \lambda dt + o(dt)$
                    $q_{0,0}(dt) = 1 - \lambda dt + o(dt)$
and at i = m

                    $q_{m,m}(dt) = 1 - m\mu dt + o(dt)$
                    $q_{m,m-1}(dt) = m\mu dt + o(dt)$

Thus the generator matrix is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & . \ldots \ldots \ldots & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \ldots & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda \ldots \ldots .0 \ldots \ldots .0 \\ . & . & & & \\ . & . & & & \\ 0 & 0 & 0 & (m\text{-}1)\mu & -(\lambda+(m\text{-}1)\mu) & \lambda \\ 0 & 0 & 0 & 0 & m\mu & -m\mu \end{pmatrix}$$

ii. Stationary distribution satisfies the equation $\underline{\pi}\, G = \underline{0}$. Thus,

$$- \lambda \pi_0 + \mu \pi_1 = 0$$

$$\lambda \pi_0 - (\lambda + \mu) \pi_1 + 2\mu \pi_2 = 0$$

.

$$\lambda \pi_k - (\lambda + (k + 1)\mu)\pi_{k+1} + (k + 2)\mu \pi_{k+2} = 0 \qquad\qquad (*)\; k = 0, ...,m - 1$$

.

$$\lambda \pi_{m-1} - m\mu\pi_m = 0.$$

So, from (*), we have

$$(n + 1)\mu\pi_{n+1} - \lambda \pi_n = n\mu\pi_n - \lambda \pi_{n-1} \qquad\qquad n = 1, 2, ...,m - 1$$

iii. Continuing from part (ii), we have

$$\pi_n = \frac{\lambda}{n\mu}\pi_{n-1} = ... = \frac{\lambda^n}{n!\mu^n}\pi_0 \qquad\qquad n \leq m - 1.$$

$$\pi_m = \frac{\lambda}{m\mu}\pi_{m-1} = ... = \frac{\lambda^m}{m!\mu^m}\pi_0$$

Also, $\sum_i \pi_i = 1$.

$$\Rightarrow \sum_{i=0}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \pi_0 = 1$$

$$\pi_0 = \frac{1}{\displaystyle\sum_{i=0}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i}$$

So,

$$\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n \left\{\sum_{i=0}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right\}^{-1}$$

$$k = \left\{\sum_{i=0}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i\right\}^{-1}$$

Total - [16]

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