The Institute of Actuaries of India

Subject CT1 – Financial Mathematics

05th November 2007

INDICATIVE SOLUTION

Solution:

Q.1 INDICATIVE SOLUTION MARKS

[Total 9]

I (a) Real rates of interest allow for future inflation

Money rates of interest ignore the effects of inflation

I(b)
- In periods of positive inflation the real rate of interest will be lower than the money rate of return
- In periods of negative inflation the real rate of interest will be higher than the money rate of return
- In periods of zero inflation the real rate of interest will be equal or greater than the money rate of return

I(c) The two factors:-
- The purpose to which the rate will be put;
- Whether the underlying data have or have not already been adjusted for inflation.

II

a) The money rate of return is i where $(1+i) = 44.4/40$
$i = 0.11$ or 11%

b) The rate of inflation is f where $(1+f) = 240/224$
$f = 0.07143$ or 7.143%

c) The net real rate of return per annum is j
where (0.70x.11-0.07143)/(1.07143) = 0.005199 or 0.
5199%
[Formula= (i x (1-t) – f) / (1+f);
where t = 30%]

Q.2.

Solution:

MARKS

[total 10]

I.

a) An annuity consists of a regular series of payments

b) Let n be a non-negative number. The value at time 0 of an annuity payable continuously between time 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$

c)
$$\bar{a}_{\overline{n}|} = \int_0^n 1.v^t dt = \int_0^n e^{-\delta t} dt = \left[-\frac{1}{\delta}e^{-\delta t}\right]_0^n$$
$$= \frac{1-e^{-\delta n}}{\delta}$$
$$= \frac{1-v^n}{\delta} \quad (\text{if } \delta \neq 0)$$

II.

$$\text{Cost of the perpetuity} = v \cdot (Ia)_{\overline{n}|} + \frac{n \cdot v^{n+1}}{i}$$
$$= v \cdot \left[\frac{\ddot{a}_{\overline{n}|} - nv^n}{i}\right] + \frac{n \cdot v^{n+1}}{i}$$
$$= \frac{a_{\overline{n}|}}{i} - \frac{nv^{n+1}}{i} + \frac{nv^{n+1}}{i}$$
$$= \frac{a_{\overline{n}|}}{i}$$

Since $i = 10.5\%$,

$$\frac{a_{\overline{n}|}}{i} = \frac{a_{\overline{n}|}}{0.105} = 77.10 \Rightarrow a_{\overline{n}|} = 8.0955, \text{ at } 10.5\%$$
$$\therefore n = 19$$

Q.3.

Solution:

MARKS

[Total 19]

I.

a) The prospective method involves finding the present values of future cash flows.

The retrospective method involves calculating the accumulated value of the initial loan less the accumulated values of the repayments to date.

b) The flat rate of interest is defined as the total interest paid over the whole transaction per unit of the initial loan, per year of the loan

II.

(a) Monthly instalment is X/12 where:

$X\ddot{a}_{25}^{(12)}@6\% = 300{,}000$

$i/d^{(12)} = 1.032211$; $\ddot{a}_{25} = 12.7834$

$X = 300{,}000/(1.032211 \times 12.7834) = 22735.599$ and

$X/12 = 1894.633$

(b) After 2 years loan outstanding is:

$X\ddot{a}_{23}^{(12)}@6\% = 1.032211 \times 12.3034 \times 22735.599 = 288735.391$

Interest required = $d^{(12)}/12 \times 288735.391 = 1398.634$

Capital repaid = 1894.633 – 1398.634 = 495.999

(c) Loan outstanding is

$X\ddot{a}_{15}^{(12)}@6\% = 1.032211 \times 9.7122 \times 22735.599$

$= 227925.282$

i. Revised instalment is Y/12 such that

$Y\ddot{a}_{15}^{(12)}@2\% \quad \ddot{a}_{15}^{(12)}@2\% = 12.8493$

$i/d^{(12)} = 1.010801$, Therefore

$Y = 227925.282/(12.8493 \times 1.010801) = 17548.797$

$Y/12 = 1462.400$

ii. Annual rate of difference in instalment is

22735.599-17548.797= 5186.801

Present value of difference is

5186.801a..15(12)@2% ,where

a..15(12)@2% = 12.8493 i/d(12)= 1.010801

Therefore present value is

5186.801 x 12.8493x 1.010801=67366.616, which represents the profit to the borrower from exercising the option to repay the loan

Solution:

Q.4 INDICATIVE SOLUTION MARKS

[Total 13]

I. Advantages of issuing bonds at a fixed price

- The government will know the price the investor will pay at the outset and so it will know the cost of borrowing money
- It will administratively be less difficult than by tender

Disadvantages of issuing bonds at a fixed price

- The investors may be willing to pay more for the bond than the set price and so and so the money could be borrowed more cheaply if a tender is used
- Insufficient investors may be prepared to pay the set price causing only part of the offer to be sold .The government may not meet its finance requirements

II. Equities –which have regular declarations of dividend. The dividends vary according to the performance of the company issuing the stock and may be zero.

Property-which carries regular payments of rent, which may be subject to regular review

Index linked bonds-which carry regular coupon payments and a final redemption payment all of which are increased in proportion to the increase in a relevant index of inflation.

III. i)

$$i = 1.127^{\frac{1}{2}} - 1 = 6.16\%$$

this comes from the equation $(1+i)^2 = 1.127$.

ii)

Using $\dfrac{1}{1+i} = \left(1 - \dfrac{d^{(4)}}{4}\right)^4$, we have:

$$i = \left(1 - \frac{0.06}{4}\right)^{-4} - 1 = 6.23\%$$

iii)

Using the formula $1+i = \left(1 + \dfrac{i^{(p)}}{p}\right)^p$, we have:

$$i = \left(1 + \frac{0.14}{\frac{1}{2}}\right)^{\frac{1}{2}} - 1 = 13.1\%$$

Solution:

Q.5 INDICATIVE SOLUTION MARKS

[Total 11]

a)

(i) Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after 5 years:

$$E(S_5) = E\left[\prod_{t=1}^{5}(1+i_t)\right]$$

$$= \prod_{t=1}^{5} E\left[(1+i_t)\right]$$

as $\{i_t\}$ are independent

$$E(S_5) = E[1+i_t]^5$$

$$E[1+i_t] = (1+E[i_t]) = 1.035$$

$$\therefore \quad E(S_5) = (1.035)^5 = 1.187686$$

$$E\left(S_5^2\right) = E\left[\prod_{t=1}^{5}(1+i_t)^2\right] = \prod_{t=1}^{5} E\left[(1+i_t)^2\right] \text{ (using independence)}$$

$$= \left(E(1+i_t)^2\right)^5 = \left(E\left[1+2i_t+i_t^2\right]\right)^5 = \left(1+2E[i_t]+E\left[i_t^2\right]\right)^5$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5$$

$$Var(S_5) = E\left(S_5^2\right) - E(S_5)^2$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5 - E[1+i_t]^{10}$$

$$E(i_t) = 0.035$$

$$Var(i_t) = 0.03^2$$

$$\therefore \quad Var(S_5) = \left(1+2\times 0.035+0.03^2+0.035^2\right)^5 - (1.035)^{10}$$

$$= 1.416534 - 1.410598$$

$$= 0.0059356$$

Mean value of the accumulation of premiums is:

$$425000E(S_5) + 425000(1.03)^5 = (425000\times 1.187686) + (425000\times 1.15927)$$

$$= 997458$$

Standard deviation is $425000SD(S_5) = 425000\times\sqrt{0.0059356} = 32743.21$

b) Investing all premiums in the risky assets is likely to be more risky because, although there may be a higher probability of the assets accumulating to more than Rs.1 million, the standard deviation would be twice as high so the probability of a large loss would be greater.

Solution:

Q.6 INDICATIVE SOLUTION MARKS

[Total 28]

I.

a) The present value of Project A is given by (working in millions)

$$NPV_A = -2 - 4v^2 + 0.9v^5\bar{a}_{\overline{20|}}$$

Evaluating these functions at 10%, we get a net present value of –Rs.0.314 million.

b) We need to evaluate the compound interest functions in the expression for the net present value above at $i=0.0938$. If we do so we get a net present value of Rs.0.00011 million, which is close to zero. So the IRR is 9.38%.

c) The equation for the present value of the cashflows arising from Project B is:

$$NPV_B = -0.1\ddot{a}_{\overline{10|}} + Xv^{10}\left[v + 2v^2 + 3v^3 + \cdots + 10v^{10}\right] = -0.1\ddot{a}_{\overline{10|}} + Xv^{10}(Ia)_{\overline{10|}}$$

This must equal zero at a rate of interest of 9.38%. So evaluating the compound interest functions at this interest rate, we get:

$$-0.1 \times 6.90373 + X \times 0.407963 \times 30.107691 = 0$$

Solving this equation gives $X=0.056206$. So the value of X is Rs.56,200.

d) If both projects are to have the same net present value at 10%, we need to solve the equation:

$$-0.1\ddot{a}_{\overline{10}|} + X v^{10} (Ia)_{\overline{10}|} = -0.31406 \quad @10\%$$

$$\Rightarrow -0.1 \times 6.7590 + X \times 0.38554 \times 29.0359 = -0.31406$$

Solving this equation for X, we find that $X \doteq 0.032323$. So the value of X is Rs.32,320.

e) We first need to find the discounted payback period for each project, so that we know when the loan is completely repaid. For Project A, we want the solution, at 7%, of the equation:

$$-2 - 4v^2 + 0.9v^5 \bar{a}_{\overline{n}|} = 0$$

where n is the number of years from time 5 to the time when the loan is repaid.
Rearranging this equation we get:

$$\bar{a}_{\overline{n}|} = 8.5614172$$

So:

$$\frac{1-v^n}{\log 1.07} = 8.5614172 \quad \Rightarrow \quad v^n = 0.420746$$

Solving this equation either by trial and error or by using logs, we find that:

$$n = \frac{\log_e 0.420746}{\log_e (1.07)^{-1}} = 12.795$$

So the project breaks even after 12.795 years of continuous payments, *ie* 17.795 years after the start of the project.
To find the accumulated value at the end of 25 years, we need to accumulate the cashflows occurring after time 17.795 at 3%.
So:

$$0.9\,\bar{s}_{\overline{7.2045072}|} = 7.226201$$

and the accumulated profit at the end of 25 years is Rs.7.226 million.

For Project B, the discounted payback period must be a whole number of years,since cashflows only occur at annual intervals. We need to find the smallest integer value of n for which:

$$-0.1\ddot{a}_{\overline{10}|} + 0.045v^{10}(Ia)_{\overline{n}|} > 0$$

at i =7% . Rearranging this, we find that we want

$$(Ia)_{\overline{n}|} > 32.852 \text{ at } 7\%.$$

Trying out some values for n , we find that:

$$(Ia)_{\overline{9}|} = 29.656 \quad \text{and } (Ia)_{\overline{10}|} = 34.739$$

So the project only becomes profitable at the moment when the last payment is received at time 20.
So the net present value of all the cashflows at 7% is:

$$-0.1\ddot{a}_{\overline{10}|} + 0.045v^{10}(Ia)_{\overline{10}|} = 0.0431592$$

So the accumulated profit at the end of 20 years will be:

$$0.0431592 \times 1.07^{20}$$

We are using 7% because the loan is not paid off until time 20 and so there will not be a positive cash balance to invest. After time 20, the cash is invested to yield 3% *pa* and so the accumulated profit at the end of 25 years will be:

$$0.0431592 \times 1.07^{20} \times 1.03^{5} = 0.1936$$

ie Rs.193,600.

a) i. The discounted payback period is the point at which net revenues from the project can be used to repay all loans necessary to finance the project outgoings accumulated with interest (or the point at which the accumulated profit of the project becomes positive for the first time).
   ii. The payback period is the point at which total net revenues are greater than total net outgoings.

b) The discounted payback period simply shows when a project became profitable in present value terms, not how profitable it is. However, this information can be useful in the decision making process. The payback period ignores interest altogether and is therefore clearly an inferior criterion. It is also possible for the discounted payback period and the payback period to be before the end of the project but for the NPV to be negative.

Q.7 INDICATIVE SOLUTION MARKS

[Total 6]

The payments can be thought of as:

210 at time 5, 210 at time 6, 210 at time 7, …, 210 at time 15

LESS the following payments:

10 at time 5, 20 at time 6, 30 at time 7, …, 110 at time 15

(i) The present value of these at time 4 is:

$$210a_{\overline{11}|} - 10(Ia)_{\overline{11}|}$$

Evaluating these, we get:

$$a_{\overline{11}|} = \frac{1-1.035^{-11}}{0.035} = 9.0016$$

$$\ddot{a}_{\overline{11}|} = \frac{1-1.035^{-11}}{0.035/1.035} = 9.3166$$

$$(Ia)_{\overline{11}|} = \frac{9.3166 - 11\times 1.035^{-11}}{0.035} = 50.9201$$

So the present value is:

$$210\times 9.0016 - 10\times 50.9201 = 1{,}381.13$$

(ii) To find the present value at time 0, we need to discount the answer to part (i) by 4 years:

$$1{,}381.13\times 1.035^{-4} = 1{,}203.57$$

(iii) To find the accumulated value at time 15, we need to accumulate the answer to (i) by 11 years:

$$1{,}381.13\times 1.035^{11} = 2{,}016.40$$

Solution

Q.8 INDICATIVE SOLUTION MARKS

[Total 4]

The risk-free force of interest can be found from the equation:
$K = Se^{4\delta}$

where K is the forward price (6000) and S is the price of the security at issue (5360).
δ=¼ log (6000/5360)=2.8199%

Let $K_{1.5}$ be the value of the forward contract on 1 July 2006,with a term of 2.5 years Then

$K_{1.5} = 5600\, e^{2.5x0.028199} = 6009.03$

So the value of the forward contract is
(6009.05-6000) $e^{- 2.5x0.028199}$=8.4178

Alternatively, the value of the forward contract on 1 July 2006 ,using the no income formula

(5600-5360x $e^{1.5x0.028199}$) =8.4178

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