Institute of Actuaries of India

Subject CT1 – Financial Mathematics

October 2014 Examination

INDICATIVE SOLUTION

Solution 1:

i. $i = 1.12^{\frac{1}{2}} - 1 = 5.83\%$ **[1]**

ii. $i = \frac{d}{1-d} = 6.38\%$ **[1]**

iii. $i = e^{12 \times 0.01} - 1 = 12.75\%$ **[1]**

iv. $i = \left(1 - \frac{d^{(4)}}{4}\right)^{-4} - 1 = 8.416\%$ **[1]**

v. $1 + i = (1 + \frac{i^{(p)}}{p})^p = \left(1 + \frac{0.12}{\frac{1}{2}}\right)^{0.5} = 11.35\%$ **[1]**

[5 Marks]

Solution 2:

(i) (a) the duration is:

$$= \frac{8000\,(v + 2v^2 + 3v^3 + \ldots + 15v^{15})}{8000\,(v + v^2 + v^3 + \ldots + v^{15})} \text{ at } 9\%$$

$$= (Ia)_{\overline{15|}} / a_{\overline{15|}} = 51.8676 / 8.0607$$

= 6.4346 years

[2]

(b) The duration is:

$$= \frac{8000\,(v + (1.09 \text{ x } 2v^2) + (1.092 \text{ x } 3v^3) + \ldots + (1.0914 \text{ x } 15v^{15}))}{8000\,(v + (1.09 \text{ x } v^2) + (1.092 \text{ x } v^3) + \ldots + (1.0914 \text{ x } v^{15}))} \text{ at } 9\%$$

$= v\,(1+2+3+\ldots +15) / v\,(1+1+1+\ldots..+1)$

= 120 / 15 = 8 years

[3]

(ii) The duration in (i) (b) is higher because the payments increase over time so that the weighting of the payments is more towards end of the series of payments.

[1]

[6 Marks]

Solution 3:

Forward price of the contract is $K_0 = (S_0 - I)\, e^{\delta T} = (91 - I)\, e^{0.09*1}$

Where I is the present value of the income expected during the contract = $3.8\, e^{-0.09*6/12}$

$\Rightarrow K_0 = (91 - 3.8\ e^{-0.09*6/12})\ e^{0.09} = 95.595$

Forward price of contract set up at time r (4 months) is

$K_r = (S_r - I^r)\ e^{\delta(T-r)} = (109 - I^r)\ e^{0.085*8/12}$

Where I^r is the present value of the income expected during the contract = 3.8 $e^{-0.085*2/12}$

$\Rightarrow K_r = (109 - 3.8\ e^{-0.085*2/12})\ e^{0.085*8/12} = 111.39$

Value of original forward contract = $(K_r - K_0)\ e^{-\delta(T-r)}$

$= (111.39 - 95.595)\ e^{-0.085*8/12}$

= ₹14.9248

[6 Marks]

Solution 4:

(i) $(1 + i_t) \sim \text{Lognormal}(\mu, \sigma^2)$

$\ln(1 + i_t) \sim N(\mu, \sigma^2)$

$\ln(1 + i_t)^{12} = \ln(1 + i_t) + \ln(1 + i_t) + ... + \ln(1 + i_t) = N(12\mu, 12\sigma^2)$

Since i_t's are independent,

$(1 + i_t)^{12} = \text{Lognormal}(12\mu, 12\sigma^2)$

$E(1 + i_t) = \exp(\mu + \sigma^2/2) = 1.09$

$\text{Var}(1 + i_t) = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1] = 0.1^2$

$0.1^2/1.09^2 = [\exp(\sigma^2) - 1] \Rightarrow \sigma^2 = 0.0083815$

$\exp(\mu + 0.0083815/2) = 1.09$

$\mu = \ln 1.09 - 0.0083815/2 = 0.081986$

$12\mu = 0.983832,\ 12\sigma^2 = 0.100578$

Assuming S_{12} being the accumulation of 1 unit in 12 years' time,

$E(S_{12}) = \exp(0.983832 + 0.100578 / 2) = 2.81263$

Expected value of investment = ₹12, 00,000 $E(S_{12})$ = ₹33, 75,156

[5]

(ii) We require $P[S_{12} < 0.9 \times 2.81263 = 2.53136]$

$P[\ln S_{12} < \ln 2.53136]$, where $\ln S_{12} \sim N(0.983832, 0.100578)$

$\Rightarrow P[N(0, 1) < (\ln 2.53136 - 0.983832) / \sqrt{(0.100578)}]$

$\Rightarrow P[N(0, 1) < -0.1736] \approx 0.43 \approx 43\%$

[3]

[8 Marks]

Solution 5:

i. $1{,}000A(2, 22) = 1{,}000[e^{\int_2^8 0.04dt} \times e^{\int_8^{20}(0.005t)dt} \times e^{\int_{20}^{22}(0.003t+0.0002t^2)dt}]$

$e^{\int_2^8 0.04dt} = 1.27125$

$e^{\int_8^{20} 0.005tdt} = 2.316$

$e^{\int_{20}^{22}(0.003t+0.0002t^2)dt} = \left[\frac{0.003t^2}{2} + \frac{0.0002t^3}{3}\right]_{20}^{22}$

$= (0.726 + 0.7098) - (0.6 + 0.5333)$

$= 1.3532$

Accumulated value = ₹1,000 X 1.27125 X 2.316X1.3532 = ₹3,984.11

[4]

ii. The effective rate of interest per annum =

$1{,}000 \times (1 + i)^{20} = 3{,}984.11$

$(1 + i) = 1.0716$

$d^{(12)} = 12[1 - (1 + i)^{-(\frac{1}{12})}]$

=0.0689

(OR)

$$3{,}984.11\left\{1-\frac{d^{(12)}}{12}\right\}^{20x12} = 1{,}000$$

$$d^{(12)} = 0.0689$$

[2]

iii. $e^{\int_{19}^{20} 0.005t\,dt} = 1.1024$

$\therefore i = 10.24\%$

[2]

iv. $v(t) = e^{-\int_0^t 0.04ds} = e^{-0.04t}$

$\rho(t) = e^{-0.03t}$

$\therefore$ Present value of paymen stream $= \int_0^4 e^{-0.03t}\, e^{-0.04t} dt = \int_0^4 e^{-0.07t}\, dt$

$= [\frac{-e^{-0.07t}}{0.07}]_0^4$

=3.489 **[3]**

[11 Marks]

<u>Solution 6</u>:

i. The price of the bonds may have fallen because interest rates have risen or because their risk has increased (for example credit risk).

[1]

ii. a) Money weighted rate of return "i" is:

$91{,}000(1+i)^5 + 860{,}000(1+i)^3 = 1{,}100{,}000$ ..

On interpolation, i = 4.65% (Exact answer is 4.652%) ...

[2]

b) $(1+i)^5 = \frac{93000}{91000} \times \frac{86000}{93000} \times \frac{1023000}{946000} \times \frac{1067000}{1023000} \times \frac{1100000}{1067000} = \frac{86000}{91000} \times \frac{1100000}{946000} = 1.0989$

$i = 1.9\%$ Per annum **[2]**

iii. **a)** Money weighted rate of return "i" is:

$$91{,}000(1+i)^5 + 93{,}000(1+i)^4 + 86{,}000(1+i)^3 + 93{,}000(1+i)^2 + 97{,}000(1+i) = 500{,}000$$

Substituting i=4.65%, LHS of above equation will be ₹527,685/-.

The above indicates that the money-weighted rate of return for the Fund manager Y is lower when compared to the Fund Manager X. **[3]**

b)

$$(1+i)^5 = \frac{93{,}000}{91{,}000} \times \frac{172{,}000}{93{,}000+93{,}000} \times \frac{279{,}000}{172{,}000+86{,}000} \times \frac{388{,}000}{279{,}000+93{,}000} \times \frac{500{,}000}{388{,}000+97{,}000}$$

$(1+i)^5 = 1.0989$

$\Rightarrow i = 1.9\%\ per\ annum.$ **[3]**

[11 Marks]

<u>Solution 7</u>:

(i) Let 't' be the discounted payback period.

Then $-720{,}000 + 84{,}000 a_{\neg t}^{(2)} = 0$ @6% p.a

$\frac{i}{i^{(2)}} a_{\neg t} = \frac{720000}{84000}$ @6%p.a

$\Rightarrow 1.014782 a_{\neg t} = 8.5714$

$a_{\neg t} = 8.4465$

$\frac{1-\left(\frac{1}{1.06}\right)^t}{0.06} = 8.4465$

$\left(\frac{1}{1.06}\right)^t = 0.49321$

$t\log\left(\frac{1}{1.06}\right) = \log 0.49321$

t=12.13yrs .

Otherwise, this can be arrived by using interpolation of a¬n factors @6% using tables.

Therefore, DPP = 12.5yrs (as annuity paid every 6months)

[4]

(ii) PV of profit =

$84{,}000\left(a^{(2)}_{\neg 12.5@6\%} + v^{12.5@6\%} a^{(2)}_{\neg 12.5@4\%}\right) - 720{,}000$.

$a^{(2)}_{\neg 12.5@6\%} = \frac{1-(0.9434)^{12.5}}{0.059126} = 8.7487$ ...

(Students can also arrive this by interpolation of factors available in actuarial formula tables- Its value is 8.7456)

$a^{(2)}_{\neg 12.5@4\%} = \frac{1-(0.96154)^{12.5}}{0.039608} = 9.7839$.

Hence PV of profit $=84{,}000(8.7487 + 0.48272 \times 9.7839) - 720{,}000$

=411, 613.07 .

Profit after 25yrs

$=411{,}613.07(1.06)^{12.5}(1.04)^{12.5}$

=1,392,288.43

[6]

[10 Marks]

Solution 8:

i) The amount of loan is

$25{,}000v + 24{,}000v^2 + \cdots \ldots + 6{,}000v^{20}$ @6% .

$= 26{,}000a_{20\neg} - 1{,}000(Ia)_{20\neg}$ @6%

$= 26{,}000 \times 11.4699 - 1{,}000 \times 98.7004$

= ₹199,517.00

[3]

ii) The 10th instalment is ₹16,000

Loan amount outstanding after 9th instalment is

$$16{,}000v + 15{,}000v^2 + \cdots \ldots + 6{,}000v^{11} \text{ @6\%} \quad \ldots$$
$$= 17{,}000a_{\overline{11}|} - 1{,}000(Ia)_{\overline{11}|} \text{ @6\%}$$
$$= 17{,}000 \times 7.8869 - 1{,}000 \times 42.7571$$
$$= ₹91{,}320.20 \ldots\ldots\ldots.$$

The interest component in the 10th instalment is:0.06X91, 320.20 =₹5,479.20
The Capital portion = ₹16,000 – ₹5,479.20= ₹10,520.80

[3]

iii) a) The capital outstanding after the payment of the 10th instalment is:

Capital o/s after the 10th instalment = 91,320.20-105, 20.80 =₹80,799.40
This will be repaid by an instalment of ₹16000/- per annum. ...

Let the remaining term be 'n' years.

$$80{,}799.40 \leq 16{,}000a_{\overline{n}|@6\%}$$

$$=> a_{\overline{n}|} \geq 5.0499$$

From tables, we can find that n will be 7 years. Therefore, the loan completes by end of 17years.
[3]

b) Let 'R' be the reduced final payment i.e., final instalment.

Then, $16{,}000a_{\overline{6}|} + Rv^7 = 80{,}799.40$ @6%.

$$R = \frac{80{,}799.40 - 16{,}000 \times 4.9173}{0.66506}$$

R= ₹3,191.60 **[3]**

c) Total interest paid during the term

=25,000+24,000+.....+16,000+6X16, 000+3191.60-199,517.60

=205,000+96,000+3,191.60-199,517.60

=₹104,674 **[2]**

[14 Marks]

Solution 9: i) Eurobonds:

- ✓ A form of medium or long-term borrowing
- ✓ Usually unsecured
- ✓ Pay regular interest payments and a final capital repayment
- ✓ Issued by large companies, governments and supra-national organisations
- ✓ Issued and traded internationally
- ✓ Often not denominated in native currency of the issuer
- ✓ Yields depend on the risk of the issuer and issue size
- ✓ Absence of full-blown government control
- ✓ Usually have novel features

[2]

ii. Preference Shares:

- ✓ A form of equity-type finance
- ✓ Offer fixed stream of investment income, if issuer makes sufficient profits
- ✓ Dividends are limited when compared with those on ordinary shares
- ✓ Preference shareholders rank above ordinary shareholders both for dividends and, usually, on winding up
- ✓ Voting rights only if dividends are unpaid or on matters having direct affects their rights
- ✓ Usually offered on cumulative basis, which means unpaid dividends are carried forward
- ✓ Less riskier than ordinary shares
- ✓ Expected return is likely to be lower than on ordinary shares
- ✓ Marketability is similar to that of loan capital

[2]

iii.

Time in years	Benchmark bond yield	Interest rate to value Eurobond	Present value factor
1	0.03	0.0458	0.95620
2	0.06	0.0766	0.86276
3	0.09	0.1074	0.73635
4	0.12	0.1382	0.59583
5	0.15	0.169	0.45806

From the above, the present value of the Eurobond is:

$$11\,(0.9562 + 0.86276 + 0.73635 + 0.59583 + 0.45806) + 100\,(0.45806)$$

$$= ₹85.5072$$

[5]

[9 Marks]

Solution 10:

The investor will receive first coupon on 30th September 2014. The net coupon per ₹100 nominal will be:

0.9 x 1.5 x (Index - March 2014 / Index – September 2012)

= 0.9 x 1.5 x 110/105

In real present value terms, this is $0.9 \times 1.5 \times (110/105) \times v \times (1+g)^{-0.5}$

Where g = 4 % per annum and v is calculated at 2.5% (per half year)

The second coupon on 31st March 2015 will be: $= 0.9 \times 1.5 \times 110/105 \times (1+g)^{-0.5}$

In real present value terms, this is $0.9 \times 1.5 \times 110/105 \times (1+g)^{-0.5} \times v^2 \times (1+g)^{-1}$

Continuing this way, the last coupon payment on 31st March 2023 will be:

$= 0.9 \times 1.5 \times 110/105 \times (1+g)^{8.5}$

In real present value terms, this is $0.9 \times 1.5 \times (110/105) \times (1+g)^{8.5} \times v^{18} \times (1+g)^{-9}$

Similarly, the real present value of redemption payment is

$$100 \times (110/105) \times (1+g)^{8.5} \times v^{18} \times (1+g)^{-9}$$

The present value of all the coupon payments and redemption payment is:

$$P = (1+g)^{-0.5} \times (110/105) \times (0.9 (v + v^2 + \ldots\ldots + v^{18}) + 100v^{18})$$

$$P = 0.98058 \times 1.04762 (0.9\, a_{\overline{18}|}2.5\% + 100v^{18}\, 2.5\%)$$

$$P = 1.02727 (0.9 \times 14.3533 + 100 \times 0.64116)$$

= ₹79.1346

[8 Marks]

Solution 11:

(i) The one-year spot rate of interest, say i_1 is 8% per annum

To calculate two-year spot rate of interest, we need to arrive at the price of the two year bond, P

$$P = 9\, a_{\overline{2}|} + 100\, v^2 \text{ at } 7\%$$

From tables,

$a_{\overline{2|}} = 1.808, \quad v^2 = 0.87344$

$\Rightarrow P = 9 \times 1.808 + 100 \times 0.87344 = 103.616$

Then the two-year spot rate of interest, i_2 is such that:

$$103.616 = 9 / (1.08) + 109 / (1+ i_2)^2$$

$\Rightarrow \quad (1+ i_2)^2 = 1.14396 \qquad \Rightarrow i_2 = 6.9562\%$

To calculate three-year spot rate of interest, we need to arrive at the price of the three year bond, Q

$Q = 9\, a_{\overline{3|}} + 100\, v^3$ at 7%

From tables,

$a_{\overline{3|}} = 2.6243, \quad v^3 = 0.8163$

$\Rightarrow Q = 9 \times 2.6243 + 100 \times 0.8163 = 105.2487$

Then the three-year spot rate of interest, i_3 is such that:

$$105.2487 = 9 / (1.08) + 9 / (1.069562)^2 + 109 / (1+ i_3)^3$$

$$105.2487 = 8.33334 + 7.86738 + 109 / (1+ i_3)^3$$

$\Rightarrow \quad (1+ i_3)^3 = 1.22406 \qquad \Rightarrow i_3 = 6.9714\%$

[8]

(ii) The one year forward rate of interest beginning from now is 8%

The forward rate for one year beginning in one year from now is $f_{1,1}$ such that:

$$1.08\,(1+ f_{1,1}) = (1.069562)^2$$

$\Rightarrow \quad f_{1,1} = 5.9224\%$

The forward rate for one year beginning in two year from now is $f_{2,1}$ such that:

$$(1.069562)^2 \ (1+ f_{2,1}) = (1.069714)^3$$

$\Rightarrow \quad f_{2,1} = 7.0018\%$

The forward rate for two year beginning in one year from now is $f_{1,2}$ such that:

$$1.08 \ (1+ f_{1,2})^2 = (1.069714)^3$$

$\Rightarrow \quad f_{1,2} = 6.4608\%$

[4]

[12 Marks]

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