Actuarial Society of India

Examinations

November 2005

ST6 – Finance and Investment B

Indicative Solutions

Q.1)

a) $U = \frac{1.5}{4}(1+e^{-0.05/4}+e^{-0.05/2})$

$= \frac{1.5}{4}(1+0.9876+0.9753)$

$= 1.1111$

[1]

$\therefore Fo = (So+U)e^{rT} = (450+1.1111)e^{0.05 X 9/12}$

$= 451.1111 \text{ X } 1.0382$

[1]

$= 468.3435$

[1]
[Total 3]

b)

risk free rate of interest	Increase
dividend yield on asset	Decrease
strange cost of cost	Increase
convenience yield on asset	Decrease

[4 x 0.5 = 2]
Total [5]

Q.2)

$P(S_5 \geq 25 / S_0 = 15)$ **[0.5]**

$= P(S_0 e^{0.6W_5 - 0.08(5)} \geq 25 / S_0 = 15)$

$= P(15e^{0.6W_5 - 0.08(5)} \geq 25)$ **[1]**

$= P(W_5 \geq 1.518)$

$= P[(N,(0,5) \geq 1.518)]$ **[1]**

$= 1 - f\left(\frac{1.518-0}{\sqrt{5}}\right)$

$= 1 - f(0.679)$

$= 0.249$ **[1.5]**

It is very difficult and almost impossible to reach this value. **[1]**

[Total 5]

Q.3)

a) Tower Law

The Tower Law states that, if the stochastic X_t has filtration F_t and $t_1 < t_2 < t_3$, then $E\{E[X_{t3}| F_{t2}]|F_{t1}\} = E\{X_{t3}|F_{t1}]$

[2]

b) Equation for the conditional expectation

We have: $E\left[B_n^2 / F_{n-1}\right] = E\left[B_{n-1} + (B_n - B_{n-1})\right\}^2 / F_{n-1}\right]$

$$= E\left[B_{n-1}^2 + 2B_{n-1}(B_n - B_{n-1}) + (B_n - B_{n-1})^2 / F_{n-1}\right]$$

$$= B_{n-1}^2 + 2B_{n-1}E[B_n - B_{n-1}] + E[(B_n - B_{n-1})^2]$$

$$= B_{n-1}^2 + 2B_{n-1}X0 + 1$$

$$= B_{n-1}^2 + 1$$

[3]

c) Derive an expression

Using the results derived in part (ii) with n= 3 and n=2, we have:

$$E\left(E\left[B_3^2 / F_2\right] / F_1\right) = E\left[B_2^2 + 1 / F_1\right]$$

$$= E\left[B_2^2 / F_1\right] + 1$$

$$= B_1^2 + 1 + 1$$

$$= B_1^2 + 2$$

[1]

d) Evaluate the expression using the Tower law

Using the Tower Law with $X_t = B_t^2 and t_1 = 1, t_2 = 2 and t_3 = 3, we have$

$$E\left(E\left[B_3^2 / F_2\right] / F_1\right) = E\left[B_2^2 / F_1\right]$$

[1]

Evaluating the expression on the right hand side in a similar way to part (b), we get:

$$E\left[B_3^2 / F_1\right] = E\left[\left(B_1 + (B_3 - B_1)\right)^2 / F_1\right]$$

$$= E\left[B_1^2 + 2B_1(B_3 - B_1) + (B_3 - B_1)^2 / F_1\right]$$

$$= B_1^2 + 2B_1 E[B_1 - B_2) + E\left[(B_1 - B_2)^2\right]$$

$$= B_1^2 + 2B_1 x0 + 2$$

$$= B_1^2 + 2$$

[2]

This gives the same answer as before. **[Total 3]**

e) Find a martingale

We can rewrite the result in (ii) in the form:

$$E\left[B_n^2 - n / F_{n-1}\right] = B_n^2 - (n-1)$$

[0.5]

If we let $f(B_t) = B_1^2 - t$, this is:

$$E[f(B_n)|F_{n-1}] = f(B_{n-1})$$

So $f(B_t) = B_1^2 - t$, is a martingale

[0.5]
[Total 1]

Q.4)

a) Value of European put option (using risk-neutral valuation)

We can use the percentages given for the price movements (2½% up, 5% down) to construct the tree of future share prices:

		84.05
	82	
80		77.90
	76	
		72.20

Our "tree" is shown here in table form, without the "branches".

Since it is at the money, the strike price of the put option must equal 80. So its payoff functions is max (80-S_T,0).

We can now add the payoffs for the put option (shown in brackets) to the tree:

		84.05 (0.00)
	82	
80		77.90 (2.10)
	76	
		72.20 (7.80)

The risk-neutral probability of an up movement is:

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.01} - 0.95}{1.025 - 0.95} = 0.800669$$

We can now apply the risk-neutral pricing formula to work out the option prices after 3 months. At the node corresponding to a share price of 82, we have:

$$V_1 = e^{-r}[0 \times q + 2.10 \times (1-q)] = 0.4144$$

At the 76 node, we have:

$$V_1 = e^{-r}[2.10 \times q + 7.80 \times (1-q)] = 2.2040$$

We can then use these prices to work out the initial price:

$$V_0 = e^{-r}[0.4144 \times q + 3.2040 \times (1-q)] = 0.9608$$

So the completed tree looks like this:

		84.05 (0.00)
	82(0.4144)	
80(0.9608)		77.90 (2.10)
	76(3.2040)	
		72.20 (7.80)

The price of the European version of this put option is 0.96. **[3]**

This price is quite low because the price movements in the question are relatively small, suggesting that this is a share with a very low volatility.

b)

i) Value of European put option (using no arbitrage)

We can arrive at the same answer using a replicating strategy.

Suppose we can currently at the 82 node. We can select a portfolio of f shares and y units of cash to replicate the possible payoffs of 0.00 and 2.10 if:

$$84.05f + ye^{0.01} = 0.00 \; and \; 77.90f + ye^{0.01} = 2.10$$

[1]

solving these gives:

$$f = \frac{0.00 - 2.10}{84.05 - 77.90} = 0.3415 \; and \; y = \frac{0.00 - 84.05f}{e^{0.01}} = 28.4144$$

ii) Since this replicating portfolio has identical payoffs to the option at the 84.05 and 77.90 nodes, it must have the same value. Otherwise , there would be an arbitrage opportunity.

$$82f + y = 0.4144$$

[1]

This agrees with our previous calculation.

Similarly, at the 76 node the replicating portfolio can be found by solving the equation:

$$77.90f + ye^{0.01} = 2.10 \; and \; 72.20f + ye^{0.01} = 2.80$$

[1]

The replicating portfolio consists of -1 shares and 79.2040 units of cash, which has a value of 3.2040, as before.

[1]

At the 80 node, we need to replicate the 3 month values, ie 0.41 and 3.20. The required replicating portfolio consists of -0.4649 shares and 38.1549 units of cash, which has a value of 0.9608, as before.

[1]

c) Value of American put option

If it is an American option, we have the additional choice of exercising the option after 3 months. **[0.5]**

If we examine the tree, we can see that at the 76 node the price we have calculated is 3.20, but we would get a payoff equal to the intrinsic value of 80-76=4, which is higher, if we exercised at that node.

[0.5]

However, we would not choose to exercise at the 82 node, since the option is out of the money there.

[0.5]

We now need to recalculate the initial price as follows, based on the figure on 4.00:

$$V_0 = e^{-r}[0.4144 \times q + 4.00 \times (1-q)] = 1.1179$$

We also have the choice of exercising at the start. However, we would not choose to do this, since the option is at the money there and has a pay off of zero.

[0.5]

So the price of the American version of this put option is 1.12.

Q.5) Using he definition of S_1 we have:

$S_1 = e^{-rx1} S_1 = e^{-0.05x1}$ X 60 = 57.074 **[1.5]**

We can also find the values of S_n at time 2. If the share price goes up to 80, then:

$S_2 = e^{-rx2} S_2 = e^{-0.05x2}$ X 80 = 72.387 **[1.5]**

If the share price goes down to 50, then:

$S_2 = e^{-rx2} S_2 = e^{-0.05x2}$ X 50 = 45.242

[1]

Thus:

$E_Q[S_2 \mid F_1]$ = 72.387 x 0.43588 + 45.242 x (1-0.43588)

= 57.074 **[1]**

= S_1

Total [5]

Q.6)

a)

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r - q + \frac{s^2}{2})(T-t)}{s\sqrt{T-t}}$$

Assuming T – t = u, we get

$$\frac{\ln(\frac{S_t}{K})}{s\sqrt{u}} + \frac{(r - q + s^2)u}{s\sqrt{u}}$$

$$d_2 = d_1 - s\sqrt{u}$$

d_2 differs with d_1 only by $s\sqrt{u}$, which is independent of S

$$\frac{dd_1}{dS_t} = \frac{dd_2}{dS_t} = \frac{1}{\frac{S_t}{K} s\sqrt{u}} \frac{1}{K} = \frac{1}{S_t s\sqrt{u}}$$

$$\frac{dd_1}{dt} = \frac{dd_1}{du}\frac{du}{dt}$$

$$\frac{dd_1}{dt} = \frac{d}{du}\left[\frac{\ln(S_t/K)}{s\sqrt{u}} + \frac{(r-q+\frac{s^2}{2})}{s}\sqrt{u}\right]\frac{d}{dt}(T-t)$$

$$= \left[-\frac{\ln(S_t/K)}{2su\sqrt{u}} + \frac{(r-q+\frac{s^2}{2})}{2s\sqrt{u}}\right](-1)$$

$$\frac{dd_1}{dt} = \left[\frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{(r-q+\frac{s^2}{2})}{2s\sqrt{u}}\right]$$

$$\frac{dd_2}{dt} = \frac{dd_1}{dt} - \frac{d}{dt}s\sqrt{u}$$

$$= \left[\frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{(r-q+\frac{s^2}{2})}{2s\sqrt{u}}\right] - \frac{s}{2\sqrt{u}}(-1)$$

$$\frac{dd_2}{dt} = \frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{r-q-\frac{s^2}{2}}{2s\sqrt{u}}$$

b)

$$\frac{f(d_1)}{f(d_2)} = \frac{\frac{1}{\sqrt{2p}}e^{-\frac{1}{2}d_1^2}}{\frac{1}{\sqrt{2p}}e^{-\frac{1}{2}(d_1 - s\sqrt{u})^2}}$$

$$= e^{-\frac{d_1^2}{2} + \frac{(d_1 - s\sqrt{u})^2}{2}}$$

$$= e^{-d_1 s\sqrt{u} + \frac{1}{2}s^2 u}$$

$$= e^{-\left[\frac{\ln(S_t/K) + (r-q+\frac{s^2}{2})u}{s\sqrt{u}}\right]s\sqrt{u} + \frac{1}{2}s^2 u}$$

$$= e^{-[\ln(S_t/K)+(r-q+\frac{s^2}{2})u]+\frac{1}{2}s^2u}$$

$$= \frac{K}{S_t} e^{-(r-q+\frac{s^2}{2})u+\frac{1}{2}s^2u}$$

$$= \frac{K}{S_t} e^{-ru} e^{qu}$$

$$\frac{f(d_1)}{f(d_2)} = \frac{K}{S_t} e^{-ru} e^{qu}$$

$$S_t e^{-qu} f(d_1) = K e^{-ru} f(d_2)$$

$$S_t e^{-q(T-t)} f(d_1) = K e^{-r(T-t)} f(d_2)$$

c)

$$c = S_t e^{-qu}\Phi(d_1) - Ke^{-ru}\Phi(d_2)$$

$$\Delta = \frac{dc}{dt} = e^{-qu}[\Phi(d_1) + S_t \frac{d\Phi(d_1)}{dS_t}] - Ke^{-ru}\frac{d\Phi(d_2)}{dS_t}$$

$$\Delta = \frac{dc}{dt} = e^{-qu}[\Phi(d_1) + S_t \frac{d\Phi(d_1)}{dd_1}\frac{dd_1}{dS_t}] - Ke^{-ru}\frac{d\Phi(d_2)}{dd_2}\frac{dd_2}{dS_t}$$

$$\frac{d\Phi(d_1)}{dd_1} = f(d_1), \ \frac{d\Phi(d_2)}{dd_2} = f(d_2)$$

From part (a), we have

$$\frac{dd_1}{dS_t} = \frac{dd_2}{dS_t} = \frac{1}{S_t s\sqrt{u}}$$

$$\Delta = e^{-qu}[\Phi(d_1) + S_t \frac{1}{S_t s\sqrt{u}} f(d_1)] - Ke^{-ru}\frac{1}{S_t s\sqrt{u}} f(d_2)$$

$$\Delta = e^{-qu}\Phi(d_1) + \frac{1}{S_t s\sqrt{u}}[S_t e^{-qu} f(d_1) - Ke^{-ru} f(d_2)]$$

From part (b), we have

$$S_t e^{-qu} f(d_1) - Ke^{-ru} f(d_2) = 0$$

$$\Delta = e^{-qu}\Phi(d_1)$$

$$\Gamma = \frac{d^2c}{dS_t^2} = \frac{d\Delta}{dS_t}$$

$$\Gamma = \frac{d}{dS_t}(e^{-qu}\Phi(d_1)$$

$$\Gamma = e^{-qu}\frac{dd_1}{dS_t}\frac{d\Phi(d_1)}{dd_1} = \frac{e^{-qu}}{S_t s\sqrt{u}} f(d_1)$$

$$\Theta = \frac{dc}{dt} = S_t[\frac{d}{du}e^{-qu}\frac{du}{dt}\Phi(d_1) + e^{-qu}\frac{d\Phi(d_1)}{dd_1}\frac{dd_1}{dt}] - K[\frac{d}{du}e^{-ru}\frac{du}{dt}\Phi(d_2) + e^{-ru}\frac{d\Phi(d_2)}{dd_2}\frac{dd_2}{dt}]$$

$$\Theta = S_t[qe^{-qu}\Phi(d_1) + e^{-qu}f(d_1)\frac{dd_1}{dt}] - K[re^{-ru}\Phi(d_2) + e^{-ru}f(d_2)\frac{dd_2}{dt}]$$

$$\Theta = qS_t e^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2) + S_t e^{-qu}f(d_1)[\frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{(r-q+\frac{s^2}{2})}{2s\sqrt{u}}] - Ke^{-ru}f(d_2)[\frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{(r-q-\frac{s^2}{2})}{2s\sqrt{u}}]$$

$$\Theta = qS_t e^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2) + [S_t e^{-qu}f(d_1) - Ke^{-ru}f(d_2)][\frac{\ln(S_t/K)}{2su\sqrt{u}} - \frac{(r-q)}{2s\sqrt{u}}] - \frac{s^2/2}{2s\sqrt{u}}[S_t e^{-qu}f(d_1) + Ke^{-ru}f(d_2)]$$

$$\Theta = -\frac{S_t e^{-qu} s\, f(d_1)}{2\sqrt{u}} + qS_t e^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2)$$

d)

$$\Theta + rS_t\Delta + \frac{1}{2}s^2 S_t^2\Gamma = -\frac{S_t e^{-qu} s f(d_1)}{2\sqrt{u}} + qS_t e^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2) + rS_t e^{-qu}\Phi(d_1) + \frac{1}{2}s^2 S_t^2 \frac{e^{-qu}}{S_t s\sqrt{u}} f(d_1)$$

$$\Theta + rS_t\Delta + \frac{1}{2}s^2 S_t^2\Gamma = qS_t e^{-qu}\Phi(d_1) - rKe^{-ru}\Phi(d_2) + rS_t e^{-qu}\Phi(d_1)$$

$$\Theta + rS_t\Delta + \frac{1}{2}s^2 S_t^2\Gamma = qS_t e^{-qu}\Phi(d_1) + r[S_t e^{-qu}\Phi(d_1) - Ke^{-ru}\Phi(d_2)]$$

$$\Theta + rS_t\Delta + \frac{1}{2}s^2 S_t^2\Gamma = qS_t e^{-qu}\Phi(d_1) + rc$$

If q = 0, we have

$$\Theta + rS_t\Delta + \frac{1}{2}s^2 S_t^2\Gamma = rc$$

Q.7)

a)

According to Put-Call parity theorem:

$p + S_t e^{-qu} = c + Ke^{-ru}$

For no-arbitrage, the above condition must hold for market prices:

$p_{mkt} + S_t e^{-qu} = c_{mkt} + Ke^{-ru}$ **(1)**

For a given volatility, the relationship must also hold for B-S prices:

$p_{BS} + S_t e^{-qu} = c_{BS} + Ke^{-ru}$ **(2)**

Subtracting (1) from (2), we get

$p_{BS} - p_{mkt} = c_{BS} - c_{mkt}$

Thus the rupee pricing error when the B-S model is used to price a European put option should exactly the same as the rupee pricing error when it is used to price the European call option with the same exercise price and time to maturity

$50 - p_{mkt} = 150 - 175$

$p_{mkt} = Rs.75$

b)

The volatilities implied by option prices in the market are called implied volatilities.
Suppose c = 1.875, S_t = 21, K=20, r = 10 and T=0.25 years
The implied volatility is the value s that, when substituted into Black-Scholes formula, gives c = 1.875.
An iterative procedure can be used to find the implied s.

c)

The Black-Scholes model assumes that implied volatilities do not vary across various strike prices and/or time maturity as s^2 pertains to S and not X. However, in practice, implied volatility is not constant and varies with change in exercise price and/or time to maturity.
Volatility smile defines the relationship between the implied volatility and strike price. Thus volatility smile defines implied volatility as a function of strike price.
Term structure of implied volatility defines the relationship between implied volatility and time to maturity of the option. Thus term structure of volatility defines implied volatility as a function of time to maturity of the option.

d)

Day	Closing Stock Price S_t	$u_t = \ln(\frac{S_t}{S_{t-1}})$	u_t^2
0	20		
1	20.10	0.004988	$2.48756\text{x}10^{-5}$
2	19.90	-0.01	0.000100
3	20.00	0.005013	$2.51256\text{x}10^{-5}$
4	20.50	0.024693	0.000610

5	20.25	-0.01227	0.000151
6	20.90	0.031594	0.000998
7	20.90	0	0
8	20.90	0	0
9	20.77	-0.00624	3.89316×10^{-5}
		$\sum_{t=1}^{9} u_t = 0.037777$	$\sum_{t=1}^{9} u_t^2 = 0.001947419$

$$s = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(u_t - \bar{u})^2} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}u_t^2 - \frac{1}{n(n-1)}(\sum_{t-1}^{n}u_t)^2}$$

$$\sqrt{\frac{0.001947419}{8} - \frac{0.037777^2}{72}} = 0.014953$$

or 1.4953% per day. There are 252 trading days per year. The data give an estimate of the volatility per annum of $0.014953\sqrt{252} = 0.23738$ or 23.74%.

Q.8)

a)

$$k = \frac{dp}{ds} = S_t\sqrt{T-t}f(d_1)e^{-q(T-t)}$$

$$d_1 = \frac{\ln(S_t/K) + (r - q + \frac{s^2}{2})(T-t)}{s\sqrt{T-t}}$$

$$d_1 = \frac{\ln(2745/2700) + (0.08 - 0.03 + \frac{0.0625}{2})\frac{1}{3}}{0.25\sqrt{\frac{1}{3}}}$$

$d_1 = 0.302157$

$$f(0.302157) = \frac{1}{\sqrt{2p}}e^{-0.302157} = 0.294907$$

$$k = 2745 * \sqrt{\frac{1}{3}} * 0.294907 * e^{-\frac{0.03}{3}} = 462.726$$

Thus a 1% (0.01) increase in volatility (from 25% to 26%) increases the value of the option by approximately 4.62726 (=0.01x462.726).

b)

NSE Nifty futures can be used to hedge an equity portfolio. Define:
P: Current value of the portfolio
A: Current value of the stocks underlying one futures contract
P = 54,000,000

A = 2700x200 = 540,000

$$N = b\frac{P}{A}$$

$$N = 1.2\frac{54{,}000{,}000}{540{,}000} = 120$$

Thus 120 Nifty futures contract should be shorted to hedge the portfolio of 54 million.
To reduce ß of portfolio from 1.2 to 0.6:

$$N = (b - b^*)\frac{P}{A}$$

$$N = (1.2 - 0.6)\frac{54{,}000{,}000}{540{,}000} = 60$$

60 Nifty futures should be shorted to reduce the beta of the portfolio from 1.2 to 0.6.

c)

Optimal hedge ratio (h) = $r\frac{s_S}{s_F}$

$$h = 0.9x\frac{1}{1.5} = 0.6$$

Each heating oil futures contract is for delivery of 40,000 gallons of heating oil. Thus, the optimal number of contracts is

$$\frac{0.6x4{,}000{,}000}{40{,}000} = 240 \text{ Contracts}$$

Jet Airways should purchase 240 heating oil futures contacts.

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