Actuarial Society of India

Examinations

November 2005

CT4(104) - Models

Indicative Solutions

Q.1

a) ${}_u q_x = \Pr\left[T_x \le u\right]$

$= \Pr\left[K_x = 0 \text{ and } S_x \le u\right] = \Pr\left[K_x = 0\right] * \Pr\left[S_x \le u\right]$

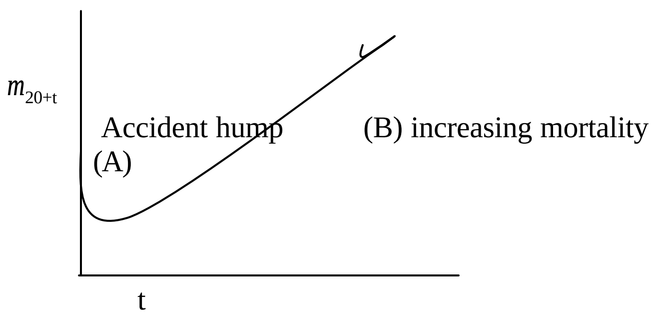
since K_x and S_x are independent.

$\Pr\left[S_x \le u\right] = \int_0^u 1dx = u$, since uniform distribution.

Thus, ${}_u q_x = u * q_x$ since $\Pr\left[K_x = 0\right] = q_x$

b)

i)



The two key components of the curve are

(A) Accident hump at around age 18

(B) Steadily increasing mortality from mid 20s.

(ii)

$\text{GM}(2,2) = a_1 + a_2 t + \exp(a_3 + a_4 t)$

$= x + yt + zc^t$ by relabelling the constants.

x is the independent of age and gives the general level of the curve

y is a linearly decreasing function, gives the ‘accident hump’.

zc^t reflects the increasing mortality, as it varies exponentially with age.

Q.2

a) ${}_{(1\text{-}t)} q_{x+t} = (1 - t) q_x$ $\quad 0 \le t < 1$

b)

We have ${}_b p_x = {}_a p_x * {}_{b-a} p_{x+a}$ for $0 \le a<b \le 1$

or

$${}_{b-a} p_{x+a} = \frac{{}_b p_x}{{}_a p_x}$$

Now

$$p_x = {}_b p_x * {}_{1-b} p_{x+b}$$

or $${}_b p_x = \frac{p_x}{{}_{1-b} p_{x+b}}$$

or

$${}_b p_x = \frac{1-q_x}{1-{}_{1-b}q_{x+b}} = \frac{1-q_x}{1-(1-b)q_x}$$ by Balducci Assumption

similarly $${}_a p_x = \frac{1-q_x}{1-(1-a)q_x}$$

Thus $${}_{b-a} p_{x+a} = \frac{{}_b p_x}{{}_a p_x} = \frac{\frac{1-q_x}{1-(1-b)q_x}}{\frac{1-q_x}{1-(1-a)q_x}}$$

$$= \frac{1-(1-a)q_x}{1-(1-b)q_x}$$

Thus $${}_{b-a} q_{x+a} = 1 - \frac{1-(1-a)q_x}{1-(1-b)q_x}$$

$$= \frac{(b-a)q_x}{1-(1-b)q_x}$$

c)

udd (Uniform Distribution of Deaths) implies a uniform distribution of death over the year of age. This implies an increasing force of mortality over the year to cancel the effect of survival probability ${}_t p_x$ and maintain a constant number of deaths. This is in contrast to the constant force of mortality over the year of age. Here the probability of dying in the last month of the year is lower than in the first month of the year because of probability of life surviving upto that point. Thus the probability of dying early is much more under constant force of mortality, considering that ${}_t q_x = q_x$ for t=1 for both assumptions.

Thus

${}_t q_x(udd) < {}_t q_x(\text{constant } m_x)$

Q.3

Assuming a Poisson distribution for deaths, we know

$Var(\hat{q}_x) = m_x E_x^C = q_x$

$\text{Std dev}(\hat{q}_x) = \sqrt{q_x}$

95% confidence interval of number of deaths = $\pm 1.96\sqrt{q_x}$ assuming Normal Distribution.

Equating actual number of death to the expected number of death, 95% CI of A/E ratio is

$$\pm 1.96\frac{\sqrt{q_x}}{q_x} = \pm\frac{1.96}{\sqrt{q_x}}$$

Q.4

a) Central exposed to risk
Period of exposure is 1.6.2000 to 25.10.2000

= 30+31+31+30+25 = 147 days
= 147/7 = 21 weeks

b) Initial exposed to risk
Period of exposure is 1.6.2000 to 31.5.2001 = 52 weeks

Q.5

a) A life alive at time t should be included in the exposure at age x at time t if and only if, were that life to die immediately, he or she would be counted in the death data dx at age x.

b)

Data allows for the following groupings

q_x = no. of deaths grouped by :

x = (calendar year of death) – (calendar year of birth) over the period 1.1.92 to 31.12.96

$P_x(n)$ = no. of policies in force at 31.12.(91+n), n= 0,1,2,3,4,5 grouped by

x = 1991 + n – (calendar year of birth)

So $P_x(n)$ = no. of policies in force at 31.12.(1991 + n) aged x last birthday at 31.12.(1991 + n)

Now x in q_x is the same as x next birthday at 1 January preceding death.

To correspond, we need the central exposed to risk defined as

$$E_x^C = \int_0^5 P_x^{'}(t)dt$$

when $P_x^{'}(t)$ = no. of lives at time t

(where t = 0 is 1.1.1992) aged x next birthday at previous 1st January.

Now $\int_0^1 P_x^{'}(t)dt = \frac{1}{2}\left(P_x^{'}(0) + P_x^{'}(1)\right)$ by Trapezium rule

assuming that $P_x^{'}(t)$ varies linearly over the calendar year.

$P_x^{'}(0)$ = no. of lives at 31.12.1991 aged x next birthday at 31.12.1991

= no. of lives at 31.12.1991 aged (x-1) last birthday at 31.12.1991

= $P_{x-1}(0)$

$P_x^{'}(1)$ = no. of lives at 31.12.92 aged x next birthday at 31.12.1991

= no. of lives at 31.12.1992 aged at x last birthday at 31.12.1992

= $P_x(1)$

Hence $\int_0^1 P_x'(t)dt = \frac{1}{2}\left(P_{x-1}(0) + P_x(1)\right)$

Over 1992 to 1996 we have this

$$E_x^C = \frac{1}{2}\sum_{n=0}^{4}\left(P_{x-1}(n) + P_x(n+1)\right)$$

This exposure will correspond to q_x defined earlier. Considering that it is a calendar year rate interval, the age range at the start of the rate interval is x-1 to x; the average age at the start of the rate interval is x-1/2; and the average age at the middle of the rate interval is x.

As the m type rate applies to the average age at the middle of the rate interval, this will apply to m_x. We are assuming that birthdays are uniformly distributed over the calendar year and also constant force of mortality between x-1/2 and x + ½.

Q.6

a)

Assuming constant force of mortality between ages 0 and 1 to be d ,

$${}_1p_0 = \exp\left[-\int_0^1 d\,ds\right] = e^{-d}$$

$= 1 - {}_1q_0 = 0.85$

Thus d=-log(0.85)=0.1625

b)

Probability of surviving 15 years = ${}_{15}p_0$

$= {}_1p_0 * {}_{14}p_1$

$= 0.85 * e^{-\int_0^{14} d\,ds} = 0.85 * e^{-14*0.02}$

$= 0.85*0.7558 = 0.6424$

Thus probability of dying in the first 15 days

$= 1 - {}_{15}p_0 = 1 - 0.6424 = 0.3576$

c)

$$e_0^0 = \int_0^\infty {}_t p_0 \, dt = \int_0^1 {}_t p_0 \, dt + \int_1^\infty {}_t p_0 \, dt$$

$$\int_0^1 {}_t p_0 \, dt = \int_0^1 e^{-dt} dt = \int_0^1 e^{-0.1625t} dt$$

$$= \frac{1}{-0.1625}\left[e^{-0.1625t} \right]_0^1$$

$$= \frac{1}{-0.1625}\left[e^{-0.1625} - 1 \right] = 0.923$$

$$\int_1^\infty {}_t p_0 \, dt = {}_1 p_0 \int_0^\infty {}_t p_1 \, dt = 0.85 \int_0^\infty {}_t p_1 \, dt$$

$$\int_0^\infty {}_t p_1 \, dt = \int_0^\infty e^{-0.02t} dt = -\frac{1}{0.02}\left[e^{-0.1625t} \right]_0^\infty$$

$$= -\frac{1}{0.02}\left[0 - 1 \right] = 50 \text{ years}$$

Thus e_0^0 = 0.923 + 0.85 * 50 = 43.423 years

d)

We have estimated the life expectation at the start of the 2nd year as in (c)

$$e_1^0 = \int_0^\infty {}_t p_1 \, dt = 50 \text{ years}$$

This is higher than the expectation at birth e_0^0

$e_0^0 < e_1^0$ because the average includes the very short lifetimes of a large proportion of insects who die in the first year itself.

Q.7

(i)

a)

Null Hypothesis Ho: The true underlying rates of withdrawal are the graduated rates.

Chi Square test

t	Z_t^2
0	11.77
1	2.69
2	5.01
3	0.45
4	1.07
5	1.62
6	2.01
7	10.11
8	0.47
9	1.88
	37.07

Degrees of freedom = 10 – 3(no.of parameters) = 7

Now $\Pr(X_7^2 \geq 20.3) = 0.005$

The tabulated value with 7 degrees of freedom at upper 5% point is 14.07.
37.07 > 14.07
Hence Ho is strongly rejected.

b)

Standardised deviation test

According to the null hypothesis, each standardized deviation should be a random event from a unit normal distribution.

t	Standardised deviation Zt
0	3.43
1	-1.64
2	-2.24
3	0.67
4	1.03
5	-1.27
6	1.42
7	3.18
8	-0.69
9	-1.37

SD	Exp No	Observed No
-∞ → -3	0	0
-3 → -2	0.2	1
-2 → -1	1.4	3
-1 → 0	3.4	1
0 → 1	3.4	1
1 → 2	1.4	2
2 → 3	0.2	0
3 → ∞	0	2

The above is far removed from a unit normal distribution. Hence reject Ho. Graduated rates clearly not a good representation of the crude rates.

c)

Sign Test

According to Ho, we would expect that the number of positive deviations to have a binomial distribution with B(10,0.5). Expected no equal to 5.
Above we observe 5 positive and 5 negative deviations. This is acceptable. Hence not enough evidence to reject Ho.

Summary Conclusion:

1. Chi squared test and individual sta ndardized deviation tests indicate that, adherence to data at many of individual durations is not acceptable. This is particularly the case at duration 0,2 and 7.
2. Satisfactory sign test indicates that the graduation runs centrally through the data.
3. Grouping of signs test if carried out would indicate if there are clumps of deviation over wide duration ranges.
4. Summing up, it appears that data cannot be adequately represented by a quadratic function. There are severe adherence problems at duration 0-2 and 6-7. Smoothness however is expected to be satisfactory as quadratic function has been fitted.

(ii)

a) A fully iterated least squares would give the same parameter values as a minimum chi square method. Hence only minor improvement can be expected.

b) Higher order polynomial would increase the flexibility of the model and will improve the adherence to data. This will however result in reduction in the level of smoothness.

c) Complete flexibility of shape on graphical graduation will sort out the problems encountered in the age ranges 0-2 and 6-7.
Care however needs to be taken over smoothness of the curve.

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