

Actuarial Society of India

Examinations

November 2005

CT1 –Financial Mathematics

Indicative Solutions

Q.1

I. Explain the “law of one price”

[2]

The “Law of One Price” states that if we assume that there are no arbitrage opportunities in a market, then it follows that any two securities or combinations of securities that give exactly the same payments must have the same price.

II. Write down the Equation of Value for an Insurance Policy

[1]

“PV premium = PV benefits + PV other outgo”

The premium for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

III. An insurance company accepts an obligation to pay Rs.10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds at a total cost of X in order to exactly match its obligation:

(i) 1-year 4% annual coupon bond with a yield rate of 5%

(ii) 2-year 6% annual coupon bond with a yield rate of 5%.

Calculate X .

[4]

Let F_1 and F_2 be face amounts of 1-year and 2-year bonds respectively

At the end of the second year,

$$F_2 + 0.06F_2 = 10000$$

$$F_2 = 9433.96$$

[1]

At the end of the first year,

$$0.06(9433.96) + F_1 + 0.04F_1 = 10000$$

$$F_1 = 9071.12$$

[1]

The price of the 1-year bond is $(1.04/1.05)(9071.12) = 8984.73$

[1]

The price of the 2-year bond is 2

$$((0.06/1.05) + (1.06/1.05^2)) * (9433.96) = 9609.38$$

The total price is $8984.73 + 9609.38 = 18,594.11$

[1]

- IV. Sania, Jaya, and Lisa each borrow Rs.5000 for five years at a nominal interest rate of 12%, compounded half-yearly. Sania has interest accumulated over the five years and pays all the interest and principal in a lump sum at the end of five years. Jaya pays interest at the end of every six-month period as it accrues and the principal at the end of five years. Lisa repays her loan with 10 level payments at the end of every six-month period. Calculate the total amount of interest paid on all three loans.

[5]

Sania

$$5000 \left(\left(1 + \frac{0.12}{2} \right)^{10} - 1 \right) = 8954.24 - 5000 = 3954.24$$

[2]

Jaya

$$5000 (0.06)(10) = 3000.00$$

[1]

Lisa

$$P(10) - 5000 = 1793.40 \text{ where } P = \frac{5000}{a_{\overline{10}|6\%}} = 679.35$$

[2]

The sum is 8747.64.

Q.2

I. Prove

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

[3]

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \dots + nv^n$$

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \dots + nv^{n-1}$$

$$\Rightarrow i(Ia)_{\overline{n}|} = (1 + v + v^2 + \dots + v^{n-1}) - nv^n$$

$$\Rightarrow (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

(1+1+1, for each step)

[3]

II. A loan is repayable by an increasing annuity payable annually in arrears for 15 years.

The repayment at the end of the first year is Rs.3,000 and subsequent payments increase by Rs.200 each year. The repayments were calculated using a rate of interest of 8% per annum effective.

a) Calculate the original amount of the loan.

[3]

b) Construct the capital/interest schedule for years nine (after the eighth payment) and ten, showing the outstanding capital at the beginning of the year, the interest element and the capital repayment.

[6]

c) Immediately after the tenth payment of interest and capital, the interest rate on

the outstanding loan is reduced to 6% per annum effective. Calculate the amount of the eleventh payment if subsequent payments continue to increase by Rs.200 each year, and the loan is to be repaid by the original date, i.e. 15 years from commencement.

[5]

[Total 17]

a)

$$\text{Loan} = 200(Ia)_{\overline{15}|} + 2800a_{\overline{15}|} @ 8\%$$

[1]

$$(Ia)_{\overline{15}|} = \frac{\ddot{a}_{\overline{15}|} - 15v^{15}}{0.08} = \frac{1.08 \times 8.5595 - 15 \times 0.31524}{0.08}$$

$$= 56.4458$$

[1]

$$\Rightarrow \text{Loan} = 200 \times 56.4458 + 2,800 \times 8.5595$$

$$= 35,255.76$$

[1]

b)

Capital o/s after 8th payment

$$= 200(Ia)_{\overline{7}|} + 4,400a_{\overline{7}|} @ 8\%$$

$$(Ia)_{\overline{7}|} = \frac{\ddot{a}_{\overline{7}|} - 7v^7}{0.08} = \frac{1.08 \times 5.2064 - 7 \times 0.58349}{0.08}$$

$$= 19.2310$$

$$\Rightarrow \text{Cap o/s} = 200 \times 19.2310 + 4,400 \times 5.2064$$

$$= 26,754.36$$

[2]

Year	Loan o/s at start	Repayment	Interest element	Capital element
9	26,754.36	4,600	2,140.35	2,459.65
10	24,294.71	4,800	1,943.58	2,856.42

[2]

[2]

c)

Loan o/s after 10th payment

$$= 24,294.71 - 2,856.42 = 21,438.29$$

[2]

Let 11th payment be X then

$$200(Ia)_{\overline{5}|} + (X - 200)a_{\overline{5}|} = 21,438.29 \text{ @ } 6\%$$

$$(Ia)_{\overline{5}|} = \frac{1.06 \times 4.2124 - 5v^5}{0.06} = 12.1476$$

$$\text{Hence } 200 \times (12.1476 - 4.2124) + X \times 4.2124 = 21,438.29$$

$$\Rightarrow X = 4,712.57$$

[2]

[1]

Q.3

On 1 April 2002 a large number of nominal Rs.1 zero-coupon bonds were issued, to be redeemed at par on 1 April 2007. The issue price was Rs.0.67 per Rs.1 bond, and the table gives the prices at 1 April for each year until redemption in 2007.

The prices are used for both the sale and purchase of bonds, and fractions of bonds may be bought and sold.

Date	1/4/2002	1/4/2003	1/4/2004	1/4/2005	1/4/2006	1/4/2007
	67p	78p	81p	86p	95p	100p

Consider two investors A and B. Investor A invests Rs.100 in bonds on 1 April each year from 2002 to 2006 inclusive; and investor B buys 100 bonds on 1 April each year from 2002 to 2006 inclusive.

For both investors construct a schedule to show the value of the fund at 1 April each year from 2002 to 2007 inclusive. From this calculate for the period from 1 April 2002 to 1 April 2007 for both investors:

- (i) the money weighted rate of return per annum
- (ii) the time weighted return per annum

[12]

Year	Price	Fund A New	Fund B New
2002	67	100	100.00
2003	78	100	216.42
2004	81	100	324.74
2005	86	100	444.79
2006	95	100	591.33
2007	100		622.46

[4]

i)

Money weighted rates of return

$$A: 100\ddot{s}_{\overline{5}|} = 622.46 \quad i \sim 7.4\%$$

$$B: 67(1+i)^5 + 78(1+i)^4 + 81(1+i)^3 + 86(1+i)^2 + 95(1+i) = 500$$

$$\text{Try } i = 8\% \quad \text{LHS} = 509.51 \quad i = 7\% \quad \text{LHS} = 495.56 \quad i \sim 7.3\%$$

[2+2]

ii)

Time weighted rates of return will all be equal to the time weighted return on the fund given by

$$67(1+i)^5 = 100 \quad \text{i.e. } i = 8.3\%$$

Otherwise:

$$A: (1+i)^5 = \frac{116.42}{100} \cdot \frac{224.74}{216.42} \cdot \frac{344.79}{324.74} \cdot \frac{491.33}{444.79} \cdot \frac{622.46}{591.33} \Rightarrow i = 8.3\%$$

$$B: (1+i)^5 = \frac{78}{67} \cdot \frac{162}{156} \cdot \frac{258}{243} \cdot \frac{380}{344} \cdot \frac{500}{475} \Rightarrow i = 8.3\%$$

Time weighted rates of return will all be equal to the time weighted return on the fund given by

$$67(1+i)^5 = 100 \quad \text{i.e. } i = 8.3\%$$

Otherwise:

$$\text{A: } (1+i)^5 = \frac{116.42}{100} \cdot \frac{224.74}{216.42} \cdot \frac{344.79}{324.74} \cdot \frac{491.33}{444.79} \cdot \frac{622.46}{591.33} \Rightarrow i = 8.3\%$$

$$\text{B: } (1+i)^5 = \frac{78}{67} \cdot \frac{162}{156} \cdot \frac{258}{243} \cdot \frac{380}{344} \cdot \frac{500}{475} \Rightarrow i = 8.3\%$$

[2+2]

Q.4

A pension fund expects to make payments of Rs.100,000 per annum at the end of each of the next five years. It wishes to immunise these liabilities by investing in two zero coupon bonds which mature in five years and in one year respectively.

The rate of interest is 5% per annum effective.

(i)

- (a) Show that the present value of the liabilities is Rs.432,948.
 (b) Show that the duration of the liabilities is 2.9 years.

[6]

(ii)

Calculate the nominal amounts of the two zero coupon bonds which must be purchased if the pension fund is to equate the present value and duration of assets and liabilities.

[6]

(iii)

- (a) Calculate the convexity of the assets.
 (b) Without calculating the convexity of the liabilities, comment on whether you think Redington's immunisation has been achieved.

[6]

[Total 18]

i)

$$(a) \quad 100,000 a_{\overline{5}|} @ 5\% \quad a_{\overline{5}|} = 4.32948$$

$$\Rightarrow PV = 432,947.667$$

[2]

$$(b) \quad \sum_{t=1}^5 t \times 100,000 \times v^t = 100,000 (Ia)_{\overline{5}|} @ 5\%$$

$$(Ia)_{\overline{5}|} = \frac{\ddot{a}_{\overline{5}|} - 5v^5}{i} = 12.5664 @ 5\%$$

$$\text{numerator} = 1,256,640$$

[2]

$$\Rightarrow \text{Duration} = 1,256,640 / 432,947.667$$

$$= 2.9$$

[2]

ii)

PV of zero coupon bond holdings is:

$$Av^5 + Bv = 432,947.667 \quad (1)$$

[2]

Duration numerator is:

$$5Av^5 + Bv = 1,256,640 \quad (2)$$

[2]

$$\Rightarrow 5Av^5 - Av^5 = 1,256,640 - 432,947.667$$

$$\Rightarrow A = \frac{823,692.333}{4v^5} \quad v^5 = 0.783526$$

$$\Rightarrow A = 262,815.89$$

[1]

Using equation (1),

$$B = \frac{432,947.667 - 262,815.89v^5}{v}$$

$$v = 0.952381 \Rightarrow B = 238,375.80$$

[1]

iii)

Convexity of assets

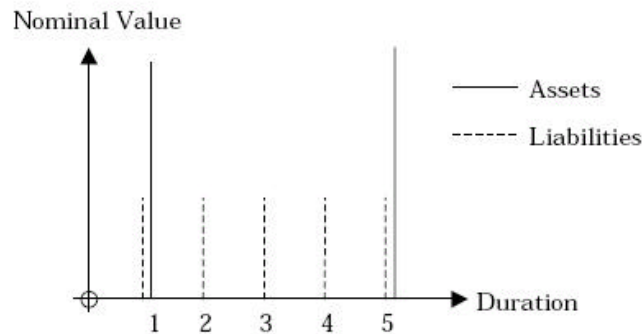
$$25Av^5 + Bv = 25 \times 262815.89 \times (1.05)^{-5} + 238375.80 (1.05)^{-1}$$

$$= 5375102.741$$

$$\text{Convexity} = \frac{(1.05)^{-2} \times 5375102.741 + (1.05)^{-2} \times 1256640}{432947.667}$$

$$= 13.89356 \quad 13.89 \text{ years}^2.$$

[4]



Assets appear to be more “spread out” than liabilities, so likely that immunisation against small changes in market rate (in Redington sense) has been achieved.

Check — Convexity of liabilities

$$100000(v + 4v^2 + 9v^2 + 16v^3 + 25v^5) = 4510643.101$$

$$\text{Convexity} = \frac{(1.05)^{-2} \{4510643.101 + 1256640\}}{432947.667}$$

$$= 12.0825 \quad 12.08 \text{ years}^2.$$

So the surmise is true.

[2]

Q.5

An investor purchased a bond with exactly 20 years to redemption. The bond, redeemable at par, has a gross redemption yield of 6%. It pays annual coupons, in arrears, of 5%. The investor does not pay tax.

(i) Calculate the purchase price paid for the bond.

[3]

(ii) After exactly ten years, immediately after payment of the coupon then due, this investor sells the bond to another investor. That investor pays

income and capital gains tax at a rate of 30%. The bond is purchased by the second investor to provide a net rate of return of 6.5% per annum.

- (a) Calculate the price paid by the second investor.
 (b) Calculate the annual effective rate of return earned by the first investor during the period for which the bond was held.

[10]

[Total 13]

$$(i) \quad P = 5a_{\overline{20}|} + 100v^{20}$$

$$= 5 \times 11.4699 + 100 \times 0.311805 = 88.53$$

[3]

$$(ii) \quad (a) \quad P' = 0.7 \times 5a_{\overline{10}|} + 100v^{10} - (100 - P') \times 0.3v^{10}$$

$$= 0.7 \times 5a_{\overline{10}|} + 70v^{10} + 0.3P'v^{10}$$

$$\therefore P' = \frac{3.5a_{\overline{10}|} + 70v^{10}}{1 - 0.3v^{10}}$$

$$a_{\overline{10}|} = 7.18883 \quad v^{10} = 0.53273$$

$$\therefore P' = \frac{62.45201}{1 - 0.159819} = 74.33161$$

[5]

$$(b) \quad 88.53 = 5a_{\overline{10}|} + 74.33161v^{10}$$

$$\text{Estimate of yield: } \frac{5}{88.53} - \frac{(88.53 - 74.33161)/10}{88.53}$$

$$= 4.04401\%$$

Return must be higher. Try 4.5%

$$a_{\overline{10}|} = 7.91272 \quad v^{10} = 0.64393$$

Price would be: 87.4279

Try 4%

$$a_{\overline{10}|} = 8.110896 \quad v^{10} = 0.67556$$

Price would be: 90.76994

$$i = -\frac{88.53 - 87.4279}{90.76994 - 87.4279} \times 0.005 + 0.045 = 0.04335$$

[5]

Q.6

- I. Explain briefly the shape of the yield curve by reference to the liquidity preference theory.

[2]

The liquidity preference theory states that, as a general rule, investors prefer to hold stocks that they are not “locked into” for a long period. They are therefore prepared to pay more for shorter, more liquid, stocks. This implies that the yields obtainable on these shorter stocks will be lower. Hence the yield curve will normally have an upward slope.

- II. An annuity pays 1 at the end of each year for n years. Using an annual effective interest rate of i , the accumulated value of the annuity at time $(n + 1)$ is 13.776. It is also known that $(1 + i)^n = 2.476$.

Calculate n

[3]

$$s_{\overline{n}|i}(1+i) = 13.776$$

$$\left[\frac{(1+i)^n - 1}{i} \right] (1+i) = \frac{2.476 - 1}{d} = 13.776$$

$$d = 0.10714 = \frac{i}{1+i}$$

$$i = 12\%$$

$$s_{\overline{n}|i} = 12.3$$

$$n = 8$$

(1/2 mark for each step = 3marks)

- III. Bhanu deposits Rs.100 into a bank account. Her account is credited interest at a nominal rate of interest i convertible semiannually. At the same time, Priya deposits Rs.100 into a separate account. Priya's account is credited interest at a force of interest of d . After 7.25 years, the value of each account is 200. Calculate $(i - \delta)$

[3]

$$\text{Bhanu} \quad \left(1 + \frac{i}{2}\right)^{(7.25)(2)} = \left(1 + \frac{i}{2}\right)^{14.5} = 2 \Rightarrow i = 9.79\%$$

[1]

Priya

$$e^{7.25\delta} = 2;$$

$$\delta = \frac{\ln 2}{7.25} = 9.56\%$$

[1]

$$\therefore (i - \delta) = 9.79\% - 9.56\% = 0.23\%$$

[1]

Q.7

- (i) State what is meant by a forward contract. Your answer should include reference to the terms .short forward position and long forward position.

[2]

- (ii) A 3-month forward contract is issued on 1 February 2001 on a stock with a price of Rs.150 per share . Dividends are received continuously and the dividend yield is 3% per annum. In addition, it is anticipated that a special dividend of Rs.30 per share will be paid on 1 April 2001.

Assuming a risk-free force of interest of 5% per annum and no arbitrage, calculate the forward price per share of the contract.

[3]

[Total 5]

(i) Definition

A forward contract is an agreement made at some time $t = 0$, say, between two parties under which one agrees to buy from the other a specified amount of an asset (denoted S) at a specified price on a specified future date. The investor agreeing to sell the asset is said to hold a short forward position in the asset, and the buyer is said to hold a long forward position.

[2]

(ii) Calculation

We know that for a security with a known dividend yield the forward price, K , is given by the formula:

$$K = S_0 e^{(\delta - D)T}$$

where S_0 is the price of the security at time 0 (in this case £150), δ is the risk-free force of interest (5% pa), D is the dividend yield (3% pa) and T is the term of the forward contract (3 months).

In this question we also have a one-off fixed dividend of £30 payable after 2 months.

The forward price of a security with fixed income is given by the formula:

$$K = (S_0 - I) e^{\delta T}$$

where I represents the present value of the fixed income payments due during the term of the forward contract.

Therefore, in order to find the forward price of the security in the question we must subtract the accumulated value of special dividend from the forward price calculated as though the security only has a known dividend yield. So:

$$K = 150 e^{(0.05 - 0.03) \frac{3}{12}} - 30 e^{0.05 \times \frac{1}{12}} \quad [2]$$

$$= 150 \times 1.0050125 - 30 \times 1.0041754$$

$$= \text{£}120.63 \text{ per share} \quad [1]$$

[3]

Q.8

- I. A company has just borrowed Rs.100,000 at a fixed rate of 8% pa. The loan is to be repaid at par value in five years and interest must be paid at the end of each year. In order to reduce its exposure to fixed rate borrowing, it arranges an interest rate swap to exchange the 8% pa fixed rate for a variable rate that is 1/2% higher than the central bank's base rate at the time when the interest payments are due.**

Over the next five years, the base rates turn out to be :

(7½%, 6½%, 7½%, 8½%, 9½%)

- a) **State the constant effective annual interest rate that would have been paid by the company if it had not arranged the swap.**
- b) **Calculate the constant effective annual interest rate that has been paid by the company as a result of taking out the swap.**

[Total 4]

- (a) Since the loan is issued at a fixed rate of 8% pa payable annually in arrears, the effective annual interest rate is 8%.

[1]

- (b) The effective annual rate is the rate that solves the equation of value:

$$100,000 = 8,000v + 7,000v^2 + 8,000v^3 + 9,000v^4 + 110,000v^5 \quad [1]$$

[1]

The interest payments are ½% higher than the declared rates.

First try 8.4% (the average of the five annual rates).

@8.4% 99,629 = RHS

@8.3% 100,030 = RHS

[1]

Therefore the effective annual rate is 8.3%.

[1]

Q.9

- I. Outline the similarities and differences between deterministic and stochastic interest rate models.**

[3]

Purpose of models

Both types are used to project the accumulated value of flows of money.

[½]

Assumptions

Deterministic models assume that future rates of return are fixed. Stochastic models assume that future rates of return are random variables.

[½]

Results obtained

For a given assumed set of future rates of return, a deterministic model will give a single definite answer. For a given assumption about the statistical distribution of future rates of return, a stochastic model will give a statistical distribution describing a range of possible answers.

[1]

Risk and uncertainty

Stochastic interest rate models make allowance for uncertainty by enabling the probability that the actual value will lie in a given range to be calculated.

Deterministic models make allowance for uncertainty by carrying out calculations

based on different sets of assumptions (eg by including contingency margins in the assumptions).

[1]

II.

The rate of interest earned in the year from time $t-1$ to t is denoted by i_t .

Assume $(1 + i_t)$ is lognormally distributed. The expected value of the rate of interest is 5%, and the standard deviation is 11%.

a) Calculate the parameters of the lognormal distribution of $(1 + i_t)$.

[4]

b) Calculate the probability that the rate of interest in the year from time $t-1$ to t lies between 4% and 7%.

[4]

[Total 11]

i)

$$(1 + i_t) \sim \log N(\mu, \sigma)$$

$$E[1 + i_t] = e^{\mu + \sigma^2/2} = 1.05 \quad \Rightarrow \quad \mu + \sigma^2/2 = 0.048790$$

[1]

$$V[1 + i_t] = (e^{\mu + \sigma^2/2})^2 (e^{\sigma^2} - 1) = (1.05)^2 (e^{\sigma^2} - 1) = 0.11^2$$

[1]

$$\Rightarrow e^{\sigma^2} = 1.010975$$

$$\Rightarrow \sigma^2 = 0.010915$$

$$\Rightarrow \mu = 0.04333, \quad \sigma = 0.104476$$

[1+1]

ii)

$$\Pr[1.04 < (1 + i_t) \leq 1.07]$$

$$= \Phi\left[\frac{\log(1.07) - \mu}{\sigma}\right] - \Phi\left[\frac{\log(1.04) - \mu}{\sigma}\right]$$

[2]

$$\begin{aligned} &= \Phi(0.233) - \Phi(-0.039) \\ &= 0.59211 - (1 - 0.51555) = 0.10766 \end{aligned}$$

[2]

*****8