Institute of Actuaries of India

Subject ST6 – Finance and Investment B

November 2013 Examinations

INDICATIVE SOLUTIONS

Solution 1 :

i. Vanilla is greater than discrete greater than continuous

(1.5 Marks)

ii. In general, it will be lower since the averaging process reduces overall variance in the risk neutral measure. **(1.5 Marks)**

iii. If the value of the sum is higher do vanilla minus the sum; otherwise do the reverse

(1.5 Marks)

iv. The asset is drift less so when the American pays off there is precisely a 50% chance (in the risk neutral measure) that the European will pay off too. So the European is worth half as much as the American. **(1.5 Marks)**

v. Price goes down **(1.5 Marks)**

vi. This follows from Put Call parity and the fact the forward with this strike has zero value.

(1.5 Marks)

vii. For p to be a risk – neutral probability, both the securities must earn an expected return equal to R, or

$$a.\, p = \frac{R-d_1}{u_1-d_1} = \frac{R-d_2}{u_2-d_2}$$

b. It is then not hard to pick R, u_1, u_2, d_1 and d_2 such that the statement for p given earlier does not hold.

(1.5 Marks)

viii. Differentiate the put call parity equation twice and the result is obvious

(1.5 Marks)

ix. Suppose there is just one period to expiration and $Sue^{-q} < X$, where q is the dividend yield per period. Clearly the option has zero value at present as it will not be exercised at expiration. However, if $S > X$, the option has positive intrinsic value, which means it should be exercised now. These two conditions imply $X < S < Xu^{-1}e^{q}$, which is possible when $u < e^{q}$.

(1.5 Marks)

x. Since X(t) is a Brownian Motion, $[X(t) - X(s)] \sim N(\mu(t-s), \sigma^2(t-s))$. Thus,

$$E[Y(t)|Y(s)] = E\left[e^{X(t)}|e^{X(s)}\right] = e^{X(s)}E\left[e^{X(t)-X(s)}\right] = Y(s)e^{(t-s)\left(\mu+\frac{\sigma^2}{2}\right)}.$$

(1.5 Marks)

[Total Marks-15]

Solution 2 :

i. The value of the trader's position is $\mathbf{V_t = \phi_t S_t C_t}$. V_t is risk free is and only if the coefficient of the randomness dW_t is zero, during the small time interval dt. The change in the value during dt is

$$dV_t = \varphi_t dS_t - dC_t = \varphi_t(\mu S_t dt + \sigma S_t dW_t) - (\mu_t^C C_t dt + \sigma_t^C C_t dW_t).$$

Since his position does not change; it is therefore risk-free if and only if

$$\varphi_t \sigma S_t = \sigma_t^C C_t.$$

(3 Marks)

ii. The condition of no arbitrage means that the drift in the no – risk position must be the short term interest rate. Thus

$$\varphi_t \mu S_t - \mu_t^C C_t = rV_t = r(\varphi S_t - C_t),$$

which, on substituting ϕ_t gives

$$\sigma_t^C \frac{C_t \mu}{\sigma} - \mu_t^C C_t = r\left(\frac{\sigma_t^C C_t}{\sigma} - C_t\right).$$

Dividing both sides by $\mathbf{C_t^C \sigma_t^C}$ gives the result. This means that there exists a common risk premium $\boldsymbol{\lambda = (\mu - r)/\sigma}$, the same for all the assets and all derivatives of it.

(3 Marks)

iii. By Ito's Lemma

$$dC_t = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2,$$

as theoher second derivatives are zero by the Ito rules. The only non-zero term is $\mathbf{(dS)^2}$ is $\mathbf{\sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S^2 dt}$. Therefore,

$$dC = \left(\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt + \sigma S \frac{\partial C}{\partial S} dW_t.$$

We then prove part c by identifying $\mathbf{C_t \mu_t^C}$ with the coefficient of dt and $\mathbf{C_t \mu_t^C}$ with the coefficient of dW_t.

(4 Marks)

iv. Tell us that

$$\mu_t^C = r + \sigma_t^C \frac{\mu - r}{\sigma} = r + \left(\frac{S}{C}\right)\frac{\partial C}{\partial S}(\mu - r).$$

Substituting in part c, we get

$$\mu^C = C^{-1}\left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) = r + \left(\frac{S}{C}\right)\frac{\partial C}{\partial S}(\mu - r).$$

Thus the result follows. This means that the derivatives can be prices without knowing the rate of return on the asset.

(4 Marks)

[Total Marks-14]

Solution 3 :

i. Moneyness is the relative position of the current price (or future price) of an underlying asset (e.g. stock) with respect to the strike price of a derivative.

First, we estimate the forward swap rate for duration 15 to 20 for the given forward bond prices. This can be done in multiple way.

If the swap is considered starting at start of 15 year (advance) then swap rate

$$\frac{\text{Bond Price (15) - Bond Price (19)}}{\text{Sum (Bond Price from 15 - 19)}} = 4\%.$$

This scenario the swaption is at money
Case when considered in advance

$$\frac{\text{Bond Price (15) - Bond Price (20)}}{\text{Sum (Bond Price from 16 - 20)}} = 5.68\%.$$

Swaption is in money as it is payer swaption and underlying swap has lower value

(3 Marks)

ii. We calculate the risk free rate at time t = 15 from the bond price to be equal to 2.9%. We can then use Black's formula

$$P_{PS} = NA[i_F\Phi(d_1) - i_S\Phi(d_2)]$$

Where

$$A = \sum_{i=1}^{n} e^{-it_i}(t_i - t_{i-1}), \quad d_1 = \frac{ln\frac{i_F}{i_S} + \frac{1}{2}\sigma_F^2 T}{\sigma_F\sqrt{T}}, \qquad d_2 = \frac{ln\frac{i_F}{i_S} - \frac{1}{2}\sigma_F^2 T}{\sigma_F\sqrt{T}} = d_1 - \sigma_F\sqrt{T}$$

Using the estimate of the market volatility to be 25%, we get:
Intermediate Calculations:

Swap Tenor (years):	5
Swap Rate Period (years):	15
D_t (Volatilty * sqrt(15)):	0.9682458
d_1:	0.4826274
d_2:	- 0.4856185
$\Phi(d_1)$:	0.6853198
Φd_2:	0.3136188
A:	4.4525463
Number of payments/ year:	1
Principal Amount:	Rs. 1000 crores
15 year forward swap rate:	4.00%
Volatility of the forward rate:	25.00%
Risk Free rate:	2.90%
Price:	Rs. 42 crores

(4 Marks)

iii. Use the implied volatility estimated from the earlier part as 25%. The new forward rate can be estimated from the bond price which is equal to 7% and the new risk free interest rate is 3.5%.

The new price can again be calculated using Black's model

D_t (Volatilty * sqrt(15)):	0.9682458
d_1:	1.002194
d_2:	0.0339488

The new price is Rs. 85 crores

(5 Marks)

iv. Market is given higher price than that can be estimated from implied volatility as the implied volatility is not the same when interest rates go up. This is referred to as the volatility smile where the underlying implied volatility changes with the moneyness of the option.

(4 Marks)
[Total Marks-16]

Solution 4 :

i. The closed form solution allows the following:

- Represent the portfolio as a basket of options
- Determine the moneyness guarantees
- Estimate the future volatility of the invested assets
- Use Black Scholes model to calculate the price of the option

(2 Marks)

ii. The broad points for the Monte – Carlo simulations are as follow (ach can be expanded further):

- Generate lots of scenarios (around 5000) and then project the balance sheet in each f these scenarios
- Estimate the cost of option in each scenario and take average of the total cost to get the price of the option. The cost is then discounted using projected yield curves.
- This require application of risk neutral pricing of underlying liability by using the principle of market consistency and arbitrage free pricing.

(4 Marks)

iii. Future risk neutral stochastic projection of interest rates, equity returns, corporate bond and property returns will be required.

- Interest rate HJM , LIBOR model, equity – constant volatility, jump diffusion model for equity, constant volatility for property
- These models need to be calibrated to the current market prices to give market consistent pricing.
- We need correlation matrix to correlate the random number used in stochastic projection.
- Key input required are
  - Interest rate model : current interest rate, swaption prices surface
  - Equity & property: option pricing surface and volatility estimates
  - We also need historical return index to estimate the correlation between each

(4 Marks)

iv. Growth fund of the asset would change according to the table below:

			Equity	
Initial Asset Mix		Actual Exposure	50%	-50%
Equity	30%	300	450	150
Corporate Bond	30%	300	300	300
Government Bond	30%	300	300	300
Property	10%	100	100	100
	100%	1000	1150	850
	% change		15%	-15%

We need to apply 15% up and -15% down for the liabilities or policy holder share.

The balance sheet becomes:

Balance sheet at t = 0		Delta 12%	
		50% Rise	50% Fall
Growth Fund	1000	1150	850
Match Fund	600	600	600
Total Asset	1600	1750	1450
Policyholders Share	1200	1380	1020
Guarantee Costs	300	278.40	321.60
Total Liability	1500	1658.40	1341.60
Free Capital	100	91.60	108.40

As can be observed from the above table for delta of 12%, free capital is lower for equity rise. This is due to fact there are 2 changes in opposite direction.

- Miss match between asset and policy holder share
- Change in guarantee due to increase in policyholder share

(6 Marks)

v. The hedging first requires a match of asset and liabilities; hence purchase equity worth Rs. 200 Crores.
Buy put options on BSE with matching delta of the guarantee

(3 Marks)

vi. Having notional positive exposure in growth fund would mean if the property goes down then policyholder share would go down due to link between the % return on growth fund and policyholder share. Over assets will not change as negative in growth fund will be compensated by positive in match fund.

The balance sheet becomes:

Balance sheet at t = 0				
		Notional Property	20% Property Price Rise	20% Property Price Fall
Growth Fund	1000	1100	1140	1060
Match Fund	600	500	480	520
Total Asset	1600	1600	1620	1580
Policyholders Share	1200	1200	1244	1156
Guarantee Costs	300	300	291	309
Total Liability	1500	1500	1535	1465
Free Capital	100	100	85	115

Suggested strategy does work. Risk is rise in the property value. Regulator may not allow this kind of transaction. **(6 Marks)**

[Total Marks-25]

Solution 5 :

The maximum value of a call option is the value of the underlying itself. This is because the call is an option to buy an asset and hence its value cannot be more than the price of the asset.

For the minimum values,

- The value has to be greater than 0
- If we construct a portfolio of one call option, short share and bond with price X/(1+r) where X is strike price, r is interest rate and S is current share price, then at expiry, we would have

	Current Value	S(t) ≤ X	S(t) > X
Call	c(0)	0	S(t) - X
Short Sell	-S(0)	-S(t)	-S(t)
Buy Bond	X/(1+r)	X	X
Total	c(0) – S(0) +{X/((1+r)^T)}	X – S(t)	0

Hence c(0) > max(0,S(0)-X/(1+r))

Similarly p(0) > max (0, X/(1+r)-S(0))

Initial Bound on American option would be current intrinsic value
C(0) ≥ Max(0,S(0)-X)
P(0) ≥ Max(0,X-S(0))

For Call option European lower bound is higher than American.

We will expect American call option to be worth more than European call option given additional feature of any time exercise. Thus lower bound of European call option holds for the American option.

Hence,

C(0) > max(0,S(0)-X/(1+r))

European put is lower than American put hence initial bound is correct.

(5 Marks)

Solution 6 :

Let the historical average from m prices be A as of time zero. The terminal payoff for a call is then

$$\max\left(\frac{mA+\sum_{i=0}^{n} S_i}{m+n+1} - X, \quad 0\right) = \max\left(\frac{\sum_{i=0}^{n} S_i}{m+n+1} - \left(X - \frac{mA}{m+n-1}\right), \quad 0\right)$$

$$= \frac{n+1}{m+n+1}\max\left(\frac{\sum_{i=0}^{n} S_i}{n+1} - \frac{m+n+1}{n+1}\left(X - \frac{mA}{m+n-1}\right), \quad 0\right)$$

So it becomes $\frac{n+1}{m+n+1}$ options with strike price $\frac{m+n+1}{n+1}\left(X - \frac{mA}{m+n-1}\right)$.

(8 Marks)

Solution 7 :

The cash flow is identical to that of the fixed rate payer / floating rate receiver. To start with, there is no initial cash flow. Furthermore, on all six payment dates, net position consists of a cash inflow computed based on \$100 million of principal, interest rate of LIBOR + 0.5% and time period of 6 month while the cash outflow is a constant \$5 million.

Hence this reflects the payments in a swap deal and thus can be represented as a synthetic swap deal.

(7 Marks)

Solution 8 :

Given that we know the dividends, we can compute the option value by replacing the current stock price S with $S(1 - \delta)^m$. This effectively implies a payoff function of

$$\max((1-\delta)^m S - X) = (1-\delta)^m max(S - (1-\delta)^{-m} X).$$

Hence proved **(5 Marks)**

Solution 9 :

The price tree takes the values 160 at time 0, 240 or 80 at time 1, 360 or 120 or 40 at time 2 and 540 or 180 or 60 or 20 at time 3. The value of p is 0.7 and that of (1 – p) = 0.3. This implies that the value of the put is 21.78.

The solutions as worked out on a excel sheet is enclosed below. Please note that the numbers in black in the working represent the asset values at different points and the numbers in red below that are the payoffs / final option value (at time t = 0).

T=0	t=1	t=2	t=3
			540.00
			0.00
		360.00	
		0.00	
	240.00		180.00
	7.47		0.00
160.00		120.00	
21.78		30.00	
	80.00		60.00
	70.00		90.00
		40.00	

		110.00	
			20.00
			130.00

160p+80	192.17
p	0.70
1-p	0.30

(5 Marks)

****