# Institute of Actuaries of India

# Subject CT8 – Financial Economics

# November 2013 Examinations

# INDICATIVE SOLUTIONS

**Introduction**

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

**Solution 1 :**

**i)** By Ito's lemma, $dZ_t/Z_t = \lambda(\mu+\sigma^2(\lambda-1)/2)dt + \lambda\sigma dW_t$

Hence $Z_t = Z_0\exp[(\mu_z-\sigma_z^2/2)t+\sigma_z W_t]$

where $\mu_z=\lambda(\mu+\sigma^2(\lambda-1)/2)$ and $\sigma_z = \lambda\sigma$

**(5 Marks)**

**ii)** $E(Z_T) = Z_0\exp(\mu_Z T)$

$= X_0^{\lambda}\exp[\lambda(\mu+\sigma^2(\lambda-1)/2)T]$

**(2 Marks)**

**iii)** We need to have $E(Z_T) = Z_0$, i.e., $(\mu+\sigma^2(\lambda-1)/2) = 0$
Alternatively, $dZ_t = \lambda X_t^{\lambda-1}dX_t + [\lambda(\lambda-1)\sigma^2/2]X_t^{\lambda-2}(dX_t)^2$

$=\lambda X_t^{\lambda-1}((\mu X_t dt + \sigma X_t dW_t)+ \lambda(\lambda-1)\sigma^2/2\ X_t^{\lambda}\ dt$

Hence $Z_t$ is a martingale if the dt term vanishes, i.e., when $2\mu=(1-\lambda)\sigma^2$

**(3 Marks)**

**[Total Marks-10]**

**Solution 2 :**

**i.** $E[N_t - N_s \mid N_s] = \lambda(t-s)$
Hence $E[N_t - \lambda t|N_s]= N_s - \lambda s$

**(1 Mark)**

**ii.** Use the fact that $\lambda(t-s) = Var(N_t-N_s) = E[(N_t-N_s)^2]- E[(N_t-N_s)]^2 = E[(N_t-N_s)^2]-\lambda^2(t-s)^2$
and $E(N_tN_s|N_s) = N_s(N_s-\lambda s+\lambda t)$ to get

$E(N_t^2|N_s) + N_s^2 -2N_s(N_s-\lambda s+\lambda t) = \lambda^2(t-s)^2 + \lambda(t-s)$

Implies $E(N_t^2|N_s) = -N_s^2 +2N_s(N_s-\lambda s+\lambda t) + \lambda^2(t-s)^2 + \lambda(t-s)$

Thus $E[(N_t-\lambda t)^2-\lambda t|N_s] = -N_s^2 +2N_s(N_s-\lambda s+\lambda t) + \lambda^2(t-s)^2 + \lambda(t-s) -2\lambda t(N_s-\lambda s+\lambda t) +\lambda^2t^2 -\lambda t$

$= (N_s-\lambda s)^2-\lambda s$

**(3 Marks)**

**iii.** The Moment generating function of Poisson(λ) distribution is given by
$M(\alpha) = E(\exp(\alpha N_t)) = \exp(\lambda(\exp(\alpha)-1))$

Given $N_s$, $N_t$-$N_s$ follows Poisson(λ(t-s).

Hence, $E[\exp(\alpha(N_t-N_s))|N_s] = \exp[\lambda(t-s)(\exp(\alpha)-1)]$

**(4 Marks)**

**[Total Marks-8]**

**Solution 3 :**

**i.** The only difference between an American and a European option is that with an American option the holder can exercise the option before the expiry date, not just on the expiry date as is the case for a European option. **(1 Mark)**

**ii.** Terminology:

$t$ is the current time
$T$ is the option expiry date
$P_t$ is the price at time t of an American put option
$p_t$ is the price at time t of a European put option
$S_t$ is the price at time t of the underlying share
$K$ is the exercise price
$r$ is the continuously compounded risk-free rate of interest

Consider a sum of money currently equal to $Ke^{-r(s-t)}$, assuming the American put option is exercised at time $s$.

Consider a portfolio comprising an American put-option on a non-dividend paying share and a share. This portfolio is worth $P_t + S_t$ at any time $t$.
At the exercise time $s$, the payoff from this portfolio is $max\{K - S_s, 0\} + S_s = max\{K, S_s\}$

Assuming no arbitrage, this portfolio at time $t$ is worth at least as much as the either the share or the discounted cash. $\Rightarrow P_t + S_t \geq S_t\ and\ P_t + S_t \geq Ke^{-r(s-t)}$
$\Rightarrow P_t \geq max\{Ke^{-r(s-t)} - S_t, 0\}$

The biggest lower bound occurs at $s = t$, which is possible for an American option as it can be exercised at any time $s \leq T$. So, $P_t \geq max\{K - S_t, 0\}$

A European option may only be exercised at time $t = T$
So, it has a weaker lower bound $p_t \geq max\{Ke^{-r(T-t)} - S_t, 0\}$

Thus, $P_t \geq p_t$

**(5 Marks)**

**[Total Marks-6]**

**Solution 4 :**

**i)** Mean return on each security:

$E(S_X) = \frac{1}{2} \times 0.04 + \frac{1}{2} \times 0.08 = 6\%$ and $E(S_Y) = \frac{1}{2} \times 0.08 + \frac{1}{2} \times 0.16 = 12\%$

Variance of return on each security:

$V(S_X) = E(S_X^2) - E^2(S_X) = \left(\frac{1}{2} \times 0.04^2 + \frac{1}{2} \times 0.08^2\right) - 0.06^2 = 0.0004$

$V(S_Y) = E(S_Y^2) - E^2(S_Y) = \left(\frac{1}{2} \times 0.08^2 + \frac{1}{2} \times 0.16^2\right) - 0.12^2 = 0.0016$

Portfolio proportion invested in security X = 2/3 and that in security Y = 1/3

Expected portfolio return $= \frac{2}{3} \times 0.06 + \frac{1}{3} \times 0.12 = 0.08$

Portfolio variance $= \left(\frac{2}{3}\right)^2 V(S_X) + \left(\frac{1}{3}\right)^2 V(S_Y) + 2 \times \frac{2}{3} \times \frac{1}{3} \times Cov(S_X S_Y)$

Coefficient of correlation = 1 $\Rightarrow Cov(S_X\, S_Y) = 1 \times \sqrt{0.0004} \times \sqrt{0.0016} = 0.0008$

$\Rightarrow$Portfolio variance $= \left(\frac{2}{3}\right)^2 0.0004 + \left(\frac{1}{3}\right)^2 0.0016 + 2 \times \frac{2}{3} \times \frac{1}{3} \times 0.008 = 0.00071$

**(4 Marks)**

**ii)** If coefficient of correlation, $Cov(S_X\, S_Y) = -1 \times \sqrt{0.0004} \times \sqrt{0.0016} = -0.0008$

$\Rightarrow$Portfolio variance $= \left(\frac{2}{3}\right)^2 0.0004 + \left(\frac{1}{3}\right)^2 0.0016 - 2 \times \frac{2}{3} \times \frac{1}{3} \times 0.008 = 0$

The expected return stays unchanged from before at 8%

**(1 Mark)**

**iii)** More correlated the investments in the portfolio; larger is the variance of return from the portfolio. Diversification across uncorrelated investments should reduce variance and hence reduce risk.

A portfolio, consisting of two perfectly negatively correlated risky assets, can be made risk-free by choosing appropriate proportions to invest in each asset.

**(2 Marks)**

**iv)**

The proportion to be invested in $S_X$ for Ganesh to have minimum variance of return of the resulting portfolio, should be $(V_Y - C_{XY}) / (V_X + V_Y - 2C_{XY})$

where, $V_X$ and $V_Y$ are the variance of return on security X and security Y respectively and $C_{XY}$ is the covariance of return.

If the coefficient of correlation = 0 $\Rightarrow$ covariance = 0

So desired $p_X = V_Y / (V_X + V_Y) = 0.0016/(0.0004 + 0.0016) = 0.8$ and $p_Y = 0.2$ **(2 Marks)**

**[Total Marks-9]**

**Solution 5 :**

ln[R(t)] = ln(RMU)+ RA.{ln[R(t - 1)] – ln(RMU)} + RBC.CE(t) + RE(t)

**i)** CE(t) = CSD.CZ(t)

RE(t) = RSD.RZ(t)

RMU, RA , RBC , CSD and RSD are parameters to be estimated

CZ(t) and RZ(t) are series of iid standard normal variables

**(2 Marks)**

**ii)** Let Y(t) be the equity dividend yield at end of year t

K(t) be the rate of dividend growth, continuously compounded, during year t

D(t) be the dividend income paid at the end of year

S(t) be the share price at time t

Then, S(t) = D(t)/Y(t) and D(t+1)/D(t) = exp[K(t+1)]

So, the total return on an equity from time t to t+1 is

$$\frac{[D(t+1)\ +\ S(t+1)]}{S(t)} - 1 = \frac{[D(t+1)\ +\ \frac{D(t+1)}{Y(t+1)}]}{\frac{D(t)}{Y(t)}} - 1$$

$$= \left[\frac{D(t+1)}{D(t)} \times \frac{[1\ +\ \frac{1}{Y(t+1)}]}{\frac{1}{Y(t)}}\right] - 1$$

$$= \left[exp[K(t+1)] \times Y(t)[1\ +\ \frac{1}{Y(t+1)}]\right] - 1$$

**(4 Marks)**
**[Total Marks-6]**

**Solution 6 :**

**i)** In the context of a single-index model of security returns, the return on a security *i* is given by

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

where $\alpha_i$ and $\beta_i$ are constants, $R_M$ is the actual return on the market and

$\varepsilon_i$ is a random variable denoting the component of return on the security that is not related to the market.

**(2 Marks)**

**ii)**

**a.** Variance of return on security *i*= $V_i$ = var[$\alpha_i + \beta_i R_M + \varepsilon_i$]

$\Rightarrow V_i$ = var[$\beta_i R_M + \varepsilon_i$] as $\alpha_i$ is a constant

Now, $\beta_i$ is also a constant and per the single index model cov($\varepsilon_i$,$R_M$) = E[($\varepsilon_i$-0)($R_M$-$E_M$)] = 0

$\Rightarrow V_i$ = var[$\beta_i R_M$]+ var[$\varepsilon_i$] = $\beta_i^2$var[$R_M$]+ var[$\varepsilon_i$] = $\beta_i^2$var[$R_M$]+ $V_{\varepsilon i}$

**b.** Covariance of returns between securities *i* and *j*,*i ≠ j* is given by$C_{ij}$= cov[$R_i$,$R_j$]

$\Rightarrow C_{ij}$ = cov[$\alpha_i + \beta_i R_M + \varepsilon_i, \alpha_j + \beta_j R_M + \varepsilon_j$] = cov[$\beta_i R_M + \varepsilon_i, \beta_j R_M + \varepsilon_j$] as $\alpha_i$ and $\beta_i$ are constants

As before, per the single index model cov($\varepsilon_i$,$R_M$) = 0
$\Rightarrow C_{ij}$ = cov[$\beta_i R_M$,$\beta_j R_M$]+ cov[$\varepsilon_i$,$\varepsilon_j$] = $\beta_i\beta_j$cov[$R_M$,$R_M$]+ 0 *(because cov[$\varepsilon_i$,$\varepsilon_j$] = 0 for i ≠ j)*
$\Rightarrow C_{ij}$ = $\beta_i\beta_j$var[$R_M$] = $\beta_i\beta_j V_M$

**(6 Marks)**
**[Total Marks-8]**

**Solution 7 :**

**i)** Ito's lemma: Let $\{X_t, t \geq 0\}$ be of the form $dX_t = Y_t dB_t + A_t dt$ where $B_t, t \geq 0$ is standard Brownian motion. Let $f: R \to R$ be twice partially differentiable wrt x and once wrt t. Then $f(t, X_t)$is also of the same form with:

$$df(t, X_t) = \frac{\partial f}{\partial x} Y_t dB_t + \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} Y_t^2\right] dt$$

**(2 marks)**

**ii)** $dX_t = \alpha dt - dW_t$ where $W_t$ is a standard Brownian motion and α is a constant

Using Ito's lemma from part (a), $A_t = \alpha$ and $Y_t = -1$
Let $f(x) = \frac{1}{x} \;\Rightarrow f'(x) = \frac{-1}{x^2} \;\Rightarrow f''(x) = \frac{2}{x^3}$

$$\Rightarrow\; df(X_t) = f'(X_t).(-1).dW_t + \left[0 + f'(X_t).\alpha + \frac{1}{2} f''(X_t) \times (-1)^2\right] dt$$
$$\Rightarrow\; d(1/X_t) = (1/X_t^2).dW_t + [\alpha \times (-1/X_t^2) + (1/X_t^3)]dt$$
$$\Rightarrow\; d(R_t) = R_t^2 dW_t + [-\alpha R_t^2 + R_t^3]dt$$

**(3 Marks)**

**iii)** A mean reverting process is one for which when the process moves away from its long-run average value, there is a factor that tends to pull it back towards the mean.

Equivalently, a mean reverting process is one with a drift which is such that the process is always attracted towards some fixed value.

From part (b), the drift coefficient of the process $R_t$ is $\left(-\alpha R_t^2 + R_t^3\right) = R_t^2\left(R_t - \alpha\right)$
So, when $R_t > \alpha$the drift coefficient is positive and the process tends to move upwards.
And when $R_t < \alpha$the drift coefficient is negative and the process tends to move downwards. Thus this process is not mean-reverting as it does not diverge towards a fixed value.

**(3 Marks)**
**[Total Marks-8]**

**Solution 8 :**

**i.** U(w) = w+αw$^2$⇒ U'(w) = 1+2αw and U''(w) = 2α

**i.** For U(w) to satisfy the requirement of risk aversion U''(w) = 2α < 0 i.e. α < 0
For U(w) to satisfy non-satiation U'(w) = 1 + 2αw > 0 i.e. -∞ < w < -1/2α (where α < 0)

**(1 Mark)**

**ii.** $A(w) = \frac{-U''(w)}{U'(w)} = \frac{-2\alpha}{1+2\alpha w}$ and $A'(w) = \frac{4\alpha^2}{(1+2\alpha w)^2} > 0$ ; thus increasing absolute risk aversion

$R(w) = w.\frac{-U''(w)}{U'(w)} = \frac{-2\alpha w}{1+2\alpha w}$and$R'(w) = \frac{-2\alpha}{1+2\alpha w} + \frac{4\alpha^2 w}{(1+2\alpha w)^2} > 0$ (because α < 0); thus implying increasing relative risk aversion

**(2 Marks)**

**iii.** The expected utility of this investor is E[U(w)] = E[w+αw$^2$] = E[w] + αE[w$^2$] which is a linear combination of only the first two moments of the distribution of wealth.

**(1 Mark)**
**[Total Marks-4]**

**Solution 9 :**

**i.** Risk-neutral probability is the probability that must be attached to an up-movement in a binomial tree model so that the expected return on the underlying share over any time interval is equal to the risk-free rate of return.

**(1 Mark)**

**ii.** Let S be the current price of the share. If $q$ is the risk-neutral probability of an upward movement, the expected final value of the share would be the same as if it were invested in risk-free cash.

So, $qSu + (1-q)Sd = Se^{r\delta t}$
$\Rightarrow qu + (1-q)d = e^{r\delta t}$
$$\Rightarrow q = \frac{e^{r\delta t} - d}{u - d}$$

The above assumes the markets are arbitrage-free else a guaranteed profit can be made.
For this, $d < u < e^{r\delta t}$ otherwise the value of q would fall outside [0,1] and thus not represent a probability.

**(5 Marks)**

**iii.** Let $p$ denote the real-world probability of an upward movement. Then, the expected real-world one-step rate of return, $\lambda$ is given by $\frac{pSu+(1-p)Sd}{S} - 1 = pu + (1-p)d - 1$

By definition, $u > d$

So, $p > q$

$\Leftrightarrow p(u-d) + d > q(u-d) + d$

$\Leftrightarrow \lambda + 1 > e^{r\delta t}$

**(3 Marks)**

**[Total Marks-9]**

**Solution 10 :**

**i.** The market price of risk is defined by $(E_M - r)/\sigma M$

Where $E_M$ = Expected return on market portfolio, r = risk free rate of return, σM = standard deviation of return from the market ……………………………

$E_M$ = 0.25*(0.2*0.4+0.3*0.2+0.5*0.1)+0.50*(0.2*0.2+0.3*0.05+0.5*0.1)+0.25*(0.2*0.1+0.3*0.2+0.5*0.07) =0.25*0.19+0.50*0.105+0.25*0.115=0.12875 = 12.875% ……………

$\sigma M^2$ = (0.25*0.4 + 0.5*0.2+0.25*0.1 – 0.12875)$^2$ * 0.2 + (0.25*0.2+0.5*0.05+0.25*0.2-0.12875)$^2$* 0.3 + (0.25*0.1+0.5*0.1+0.25*0.07 – 0.12875)$^2$* 0.5 = 0.0025 = 5%^2

Market Price =( 0.12875-0.05)/0.05

**(3 Marks)**

**ii.** The security market line is given by $E_i - r = \text{Beta}_i(E_M - r)$. The market risk if measured by Beta.

For security A, $\text{Beta}_i = (E_i - r)/(E_M - r)$ = (0.19-0.05)/(0.12875-0.05) = 0.14/0.07875=1.78

For security B = (0.105-0.05)/ 0.07875=0.7

For Security C = (0.115-0.05)/ 0.07875 = 0.825

Security A has highest market risk

**(2 Marks)**

**iii.**

$\sigma_A^2 = 129; \sigma_B^2 = 27.25; \sigma_C^2 = 32.25$

$$\sigma^2(e_A) = \sigma_A^2 - \beta_A^2\sigma_M^2 = 129 - 1.78^2 \times 25 = 49.79$$

$$\sigma^2(e_B) = \sigma_B^2 - \beta_B^2\sigma_M^2 = 27.25 - 0.7^2 \times 25 = 15$$

$$\sigma^2(e_C) = \sigma_C^2 - \beta_A^2\sigma_M^2 = 32.25 - 0.825^2 \times 25 = 15.23$$

Stock A has the highest unsystematic risk.

**(2 Marks)**

**iv.** Variance of the market portfolio due to systematic component= $\beta_M^2\sigma_M^2 = 25^2$

Variance of market portfolio due to unsystematic component = $\sigma^2(e_A) = \sigma_M^2 - \beta_M^2\sigma_M^2 = 0$

Market portfolio is well diversified.

**(3 Marks)**

**[Total Marks-10]**

**Solution 11 :**

**i.** A 3-year zero coupon bond with face value Rs. 100 will sell today at a yield of 6% and a price of:

$100/1.06^3$ =Rs. 83.96

Next year, the bond will have a two-year maturity, and therefore a yield of 6% (from next year's forecasted yield curve). The price will be Rs. 89.00, resulting in a holding period return of 6%.

**(2 Marks)**

**ii.** A 2-year zero coupon bond with face value Rs.100 will sell today at a yield of 5% and a price of:

$100/1.05^2$ =Rs. 90.70

Next year, the bond will have a one-year maturity, and therefore a yield of 5% (from next year's forecasted yield curve). The price will be Rs. 95.24, resulting in a holding period return of 5%. **(2 Marks)**

**iii.** Current price of 3-year coupon bond = $\frac{8000}{1.04} + \frac{8000}{1.05^2} + \frac{108000}{1.06^3} = 105627.40$

Expected price of bond after one year =$\frac{8000}{1.05} + \frac{108000}{1.06^2} = 103738.70$

Expected return over the next year = $\frac{8000+103738.70-105627.40}{105627.40} = 5.79\%$

**(3 Marks)**

**[Total Marks-7]**

**Solution 12 :**

**i.**

| Ratings | AAA | AA | A | BB | B | Default |
|---|---|---|---|---|---|---|
| AAA | 95% | 5% | 0% | 0% | 0% | 0% |
| AA | 5% | 85% | 10% | 0% | 0% | 0% |
| A | 1% | 2% | 85% | 7% | 5% | 0% |
| BB | 0% | 0% | 2% | 80% | 11% | 7% |
| B | 0% | 0% | 0% | 5% | 70% | 25% |

**(2 Marks)**

**ii.** A – B and B – Default. Probability 5%*25%=1.25% **(3 Marks)**

**iii.** Value of bond A before default after 1 year = 100,00*[1+exp(-0.09)+exp(-2*0.095)+

exp(-3*0.10)+exp(-4*0.105)+100,000*exp(-4*0.105)

=107,092.22
Value of bond A if it defaults to bond B after 1 year
=100,00*[1+exp(-0.10)+exp(-2*0.105)+exp(-3*0.1075)+exp(-4*0.11)+100,000*exp(-4*0.11)
=105,241,5821
Value of defaulted bond B = 0
Hence value at risk after 1 year = value of A+value of B – (value of A defaulted to B)-Value of defaulted bond
= Value of bond A.
This is the maximum loss that will be incurred in one year at 98.75% confidence level. Hence we can expect that the maximum loss at 95% confidence level will be lower than this.

Assumption: face value of bond = INR 100,000 **(7 Marks)**
**[Total Marks-12]**

**Solution 13 :**

Let $W_1$and $W_2$ quantities of option 1 and option 2 respectively are added to the portfolio to make the portfolio Gamma and Vega neutral

-10,000+1.0 $W_1$ + 1.6$W_2$ = 0
-4000+1.0 $W_1$ + 0.6$W_2$ = 0
$W_1 = 400; W_2 = 6000$
The portfolio can be made gamma and vega neutral by including 400 of option 1 and 6000 of option 2.
Delta of the portfolio after the addition of the two traded options = -1000+0.70*400+0.60*6000 =2880
Hence, 2880 units of the asset would have to be sold to make the portfolio delta neutral.

**(3 Marks)**

*****************************************