Institute of Actuaries of India

Subject ST6 – Finance and Investment B

November 2012 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable

1 **i.** Consider the process,

$X_t = B_{2t} - B_t, where\ B_t is\ a\ standard\ brownian\ motion, 0 \le t < \infty.$

Is it a Gaussian process? Can you find the mean and variance? Is it Brownian motion? **(3)**

ii. Let X_t and Y_t be independent Brownian motions. Let $Z_t = (X_t + Y_t)/\sqrt{2}$. Is Z_t a Gaussian process? Is it a Brownian motion? **(3)**

iii. Consider two non-dividend paying stocks $X_t$and Y_t. There is a single source of uncertainty which is captured by a standard Brownian motion $\{B_t\ ,\ t \ge 0\}$. The prices of the assets satisfy the stochastic differential equations:

$\frac{dX}{X} = 0.08dt + 0.15\ dB$

$\frac{dY}{Y} = \alpha dt + 0.08\ dB$

The continuously compounded risk free interest rate is 0.06. Determine α. **(3)**

2 **i.** An at the money American call option with a term of 6 month is to be written on a dividend paying stock where the ex-dividend dates are 2 months and 5 months away from now. The dividend on each ex-dividend date is expected to be Rs 0.5. The current share price is Rs 40 and the risk-free rate is 9% per annum compounded continuously. Calculate the price of the option using Black's approximation. Explain whether the answer given by Black's approach understate or overstate the true option value. **(7)**

ii. It was decided to use delta hedge to manage this option. Draw the graph of the likely movement of the delta with the passage of time for at the money, in the money and out of money call option. **(2)**

iii. A mutual fund announces that the salaries of its fund managers will depend on the performance of the fund. If the fund loses money, the salaries would be zero and if it makes a profit then the salaries will be proportional to the profit. Describe the salary as a derivative. How is a fund manager motivated to behave with this type of remuneration package? **(3)**

3 **i.** An American call option on the futures contract is traded along with an American call option on the underlying currency USD where all have the same maturity. Please argue whether the prices of both the options will be equal or different. **(3)**

ii. The exchange rate currently on USD is INR 55. There is expected to be sovereign rating announcement tomorrow which may lead the exchange rate either to 48 or to 62. What are the problems in using Black Scholes to value one month European call option on the exchange rate of USD? **3)**

4 **i.** Let $S_t$represent the stock price at time t for a non-dividend paying stock. Show that the price of the call option with exercise price of K at time T can be written as follows:

$c = e^{-rT} \int_{S_T=K}^{\infty} (S_T - K)\ f(S_T)\ dS_T$ where r is the constant risk free interest rate. **(2)**

ii. Show that $f(K) = e^{rT} \frac{\partial^2 c}{\partial K^2}$ **(3)**

iii. Explain how you can estimate the risk neutral probability distribution from observed prices on traded option. **(3)**

5 Consider a two period binomial model with t=0,1,2. There are a stock and a risk free asset. The initial stock price is 4 and the stock price doubles with probability 2/3 drops to half with probability 1/3 each period. The risk free rate is 25% per period compounded per period.

i. Compute the risk neutral probability at each node **(2)**

ii. Compute the discrete world equivalent of Radon Nikodym derivative of the risk neutral measure with respect to the physical measure at each node **(2)**

iii. Price an european lookback option with payoff at t=2 equal to $Max\{0, (max_{0\le t\le 2} S_t) - S_2\}$ using risk-neutral probability **(3)**

iv. Calculate the above price using replicating portfolio method and show that it is same as above **(4)**

v. Use European call as control variate to calculate a better approximation of the above calculated price. **(3)**

vi. Estimate delta and theta for the above derivative price at t=0. **(2)**

6 A bond dealer provides the following selected information on a portfolio of fixed income securities

Par Value(lakh)	Market Price	Coupon	Modified Duration	Effective duration	effective Convextity
500	100	6.75%	8.5	8.5	145
250	94.3	5.25%	6	1.75	45
300	93	7%	8.5	8.5	125
100	105	8%	9	4.5	-65

i. What is the effective duration for the portfolio and calculate change in value of the portfolio for a basis point change in the yield? **(3)**

ii. Which of the bond(s) most likely to has(have) no embedded options and why? **(2)**

iii. Which bonds(s) is (are) likely callable and puttable and why? **(3)**

iv. What is the approximate price change for the 7% bond if its yield to maturity increases by **(2)**

25 basis points?

v. Why might portfolio effective duration be an inadequate measure of interest rate risk for a bond portfolio even if we assume the bond effective duration are correct? **(2)**

7 An Insurance company has net equity exposure of 100 Cr, asset liability management team has suggested change in asset allocation as specified in the table

	Current	Future
Indian equity	50%	80%
UK equity	25%	10%
US equity	25%	10%

The company has decided to use stock future for implementing this transaction and aim to finish the stock allocation in 3 months.

i. State advantage of using stock futures compared to direct sale of asset in the market. **[3]**

ii. What stock futures should company buy and sell to attain the required asset allocation? Calculate value of fixed leg of each futures transaction where spot USD/INR – 55 and GBP/INR – 88. Use the following information:

- Futures on UK equities are index linked with FTSE 100 underlying priced 5450 GBP as of today(transaction date)
- Futures on US equities are index linked with Dow Jones underlying priced 13600 USD as of today(transaction date)
- Futures on Indian equities are index linked with BSE underlying priced 15000 RS as of today(transaction date)
- 3 month interest rate India – 3% UK/US =1% **[4]**

iii. Design a hedge to be implemented at the start of future contract to limit the losses due to currency movement. Also explain how will you account for interest rate differential in different economies. **[4]**

iv. The market after 3 months is as follows:

Indian interest rate	5%
UK interest rate	10%
US interest rate	5%
USD/INR spot rate	55
GBP/INR spot rate	90

Calculate net profit and loss of the company in following scenarios

(a) No equity futures (assume same number of indexes are sold/bought as future contracts on date of exchange)

(b) Equity futures with NO currency hedge **(8)**

(c) Equity futures with currency hedge

8 **i.** Define cap. How many options would you have in 2 year cap resetting quarterly? **(2)**

ii. Suppose you have purchased a 2 year cap annual reset and a strike rate of 5.0% on a notional principal of Rs 25 Cr. Assume current interest rate to be 6% and each year it can either go up 1.2 times or go down 0.8 times. Use binomial model to calculate price of all the options in this cap and price of cap itself. (1.2 times of 6% means 7.2%) **(8)**

9 **i.** How would you price a bond with maturity T in arbitrage free market and risk neutral valuation. (assume payment of d if default or 1 if not). Make simplified assumptions to write this price in terms of risk free rate and default probability p. **(4)**

ii. Suppose probability of default before time T for this bond is given by random variable $\varphi(-Z)$ where φ represents the cumulative probability distribution for a standard normal and Z is a standard normal random variable. Assume term of the bond to be 5 years, risk free rate of 6% (pa compounded continuously) and recovery rate to be 40%. Use monte carlo method (5 simulations) to calculate the credit spread over risk free rate for this corporate bond. The following are five random numbers from uniform(0,1) distribution (0.985,0.707,0.447,0.051,0.783). **(4)**

[Total Marks – 8]

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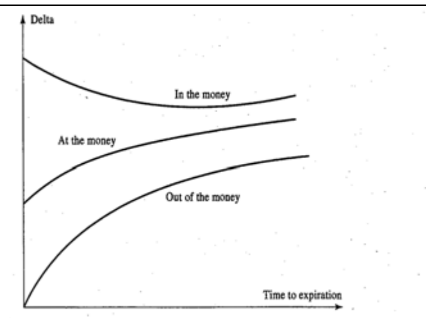
1	a	Yes, $X_t$is a Gaussian process. We must check that for each $(t_1, t_2 \dots \dots t_n)$ that $(X_{t_1}, X_{t_2}, \dots \dots X_{t_n})$ is multivariate Gaussian. That is, we need to show that for any $\theta_1, \theta_2 \dots \dots \theta_n$ the sum $S = \theta_1 X_{t_1} + \theta_2 X_{t_2} + \dots + \theta_n X_{t_n}$ is univariate Gaussian. Since S can be written as a linear combination of values of Brownian motion at various times and since Brownian motion is a Gaussian process, we see the S is indeed Gaussian. Next, we trivially have E(X_t) = 0 and Var(X_t) = t for all t. Finally, since Cov(X_s;X_t) equals $E[(B_{2s} - B_s)(B_{2t} - B_t)] = E[B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t]$ = 3min(s,t) – min(2t,s)-min(2s,t). For s=2 and t=3 this is 1 but the corresponding covariance for Brownian motion is 2. Hence , X_t is not Brownian motion.	
	b	It is easy to prove that Z is Gaussian with mean 0 and variance t. Besides, for s<=t, we have: $Cov(Z_s, Z_t) = E(Z_s Z_t) - E(Z_s)E(Z_t) = E(Z_s Z_t) = \left(\frac{1}{2}\right) E(X_s X_t + X_s Y_t + Y_s X_t + Y_s Y_t) = \left(\frac{1}{2}\right)(s + 0 + 0 + s) = s$ By generality of argument if t<=s then the answer above would have been t. Therefore, $Cov(Z_s, Z_t) = \min(s, t)$. Therefore$Z_t$is a Brownian motion.	
	c	Under no arbitrage principles the market price of risk would exist and would be same for both the securities. Therefore, $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$ $i.e. \frac{0.08 - 0.06}{0.15} = \frac{\alpha - 0.06}{0.08}$ $i.e. \alpha = 7.067\%$	
2	a	Black's approximation suggests that we calculate the price of the option at various time points before expiry treating it as a European option and then take the maximum of all such values. It is already known that call option will not be exercised for a stock with no dividend and for those paying dividends it may be exercised just before the payout of the dividend. Therefore, we need to calculate the value of the option just before the dividend payouts. The dividends D_1 and D_2 are 0.5 each. The first dividend is after 2 months where it may be exercised. Value of the interest earned on exercise price assuming exercise may happen at 1st dividend for the period between the two dividends is $40*(1 - e^{-0.09*0.25}) = 0.89$. Since the loss in the interest is more than the dividend (0.5) it is not going to be exercised before the 2nd dividend payout. Also, the loss of interest at the point of 2nd dividend is $40*(1 - e^{-0.09/12})$ = 0.30 which is less than 0.50. Hence, there exists a possibility that it may get exercised at the time of second dividend. We use black-scholes to calculate the value of the option assuming it is a European for the full term and for the term of 5 months and then take the maximum.	

		For full term: Present value of the dividends = 0.9741, S_0=40-0.9741, K= 40, r=0.09, sigma = 0.3, T= 0.5 then d_1=0.2017, d_2=-0.0104, N(d_1) = 0.58, N(d_2)= 0.4959 and c = 3.67 For 5 months everything else remains the same except T=0.4167 and S_0 = 40-0.4926 which gives c=3.52. Therefore, the price of the option according to Black's approximation is max(3.52,3.67) = 3.67. Black's approach in effect assumes that the holder of the option must decide at time zero whether it is a European option maturing at time t_n or at time T. In fact, the holder of the option has more flexibility than this. Even if the option is not exercised at time t_n he can still hold on to that option till its expiry. It appears that this may understate the true value. However, the way volatility is applied may lead to overstate the value. Black applies volatility to the stock price though the binomial model would apply the volatility to the stock price less the present value of the dividend.	
2	b		
2	c	Suppose K is the value of the fund at the beginning of the year and S_T be the value of the fund at the end of the year. Then the salary of the fund manager is αMax(0,S_T-K) where α is the constant proportion to be paid out as salary. This shows that the fund manager has a call option on the value of the fund at the end of the year. All of the parameters determining the value of this call is outside the control of the fund manager except the volatility of the fund. The fund manager has an incentive to make this fund as volatile as possible which may not be in the interest of the investors. It might be argued that the fund manager may not do this because his total income is dependent on this and would become risky for him to do this. However, he/she can hedge this risk in the market by trading on his own account for e.g. shorting call options. The strategy can be designed in such a way that if market goes up the gains from the salary would be far higher than the losses from the call options and in case of a fall, the fund manager still receives the option premium. The payoff from this strategy is positively correlated with the increase in the risk of the portfolio and incentivizes to increase the risk.	
3	a	It is generally not true that an American futures option is worth the same as the corresponding American spot option when the futures and options contract have the same maturity. In this case, though the futures price would converge to the spot price, it is likely	

		that futures would be traded at a higher rate than the spot rate before the maturity given the differences in the interest rates. Therefore, in some situations it is possible that the future option may be exercised early though spot may not. This implies that the option on futures may be worth more than the option on the spot. The difference is likely to be more when the underlying future contract expires far after the expiry of the option contract.	
	b	The probability distribution of the stock price in one month cannot be assumed to be lognormal. Possibly it may be modeled two lognormal distributions super imposition upon each other and is bimodal. Black Scholes is clearly inappropriate as it assumes lognormal over the full term and does not allow for any jumps.	
4	a	The price of a call option can be calculated using risk neutral valuation approach which suggests that $c = e^{-rT}E_Q[Max(0, S_T - K)]$ where r is the risk free interest rate, T is the term of the contract, K is the exercise price and S_T is the price of underlying stock at maturity of the option. Assuming f as the risk neutral probability density function of random variable S_T the above may be written as $c = e^{-rT}\int_{S_T=K}^{\infty}(S_T - K)\,f(S_T)\,dS_T$	
	b	Differentiating c with respect to K will give us $\frac{\partial c}{\partial K} = -e^{-rT}\int_{S_T=K}^{\infty} f(S_T)\,dS_T$ Differentiating again wrt K would give us $\frac{\partial^2 c}{\partial K^2} = -e^{-rT}f(K)$ Rearranging we get the required equation. $f(K) = e^{rT}\frac{\partial^2 c}{\partial K^2}$	
	c	We can use the traded prices at various exercise prices to estimate the second derivative in the above equation and use the equation to estimate the risk neutral density/mass function. Suppose that c_1, c_2 and c_3 are the traded prices of T year European call option with strike prices of $K-\delta, K$ and $K+\delta$ respectively then an estimate of $f(K) = e^{rT}\frac{c_1+c_3-2c_2}{\delta^2}$ assuming $\delta$is small. Even if δ is large this still gives a good approximation to estimate the risk neutral probabilities.	
5	a	Let q denote the risk-neutral probability of up-node and 1 − q denote risk-neutral probability of the down-node. Then by the definition of the risk-neutral probabilities $S_t = E_Q\left[\frac{S_{t+1}}{1+r}\right] = \frac{1}{1+r}\left[q2S_t + \frac{(1-q)}{2}S_t\right]$ where r = ¼ , this gives the q = ½ . It may be noted that this is true for all the nodes since the stock price movement has the identical grid (double up, half down).	

b The real world measure assigns probabilities for up 2/3 and for down 1/3 where as the risk neutral measure assigns equal probabilities to both the states. Radon Nikodym derivatives (dQ/dP) is a random variable which takes the value for up node (1/2)/(2/3) = ¾ and for down node it takes the value 3/2 = (1/2)/(2/3). Again note that this is true for all the nodes since the probabilities are identical for all the nodes.

c The following binomial tree describes the evaluation of the stock price and the bold face numbers next to the stock price are the payoffs from the lookback option:

```
                    16 (0)
        8           4 (4)
4
        2           4 (0)
                    1 (3)
```

The probability for each node is ¼ (½ * ½) since up and down both have same risk neutral probabilities. Therefore the value of this option is = (4/5)^2 *(1/4)*(0+4+0+3) = 28/25

d We need to create the replicating portfolios for the above option. Let x be the amount of stock and y be the amount of bond that needs to be hold at any node, then $S_ux+y(1+r) = c_u$ and $S_dx+y(1+r) = c_d$. Therefore $x = (c_u-c_d)/(S_u-S_d)$. Applying this to time 1 node at 8, we get x(8) = (0-4)/(16-4) = -1/3. Therefore one needs to short 1/3 unit of the stock for every option as at node 1. The number of bonds at this node will be = (0 + 16/3)*4/5 = 64/15 and hence the value of the option would be 64/15 – 8/3 = 8/5. It may be noted that by construction this would match the derivative values at time 2 from the node at 8.

Similarly at time 1, node at 2 we get x = (0-3)/(4-1) = -1 and y = 4*4/5=16/5 and hence value of derivative at this node = 16/5 – 2 = 6/5.

Now at time 0 the value of x = (8/5 – 6/5)/(8-2) = 1/15 and y = (8/5 – 8/15)*4/5 = 64/75 giving the value of the derivative to be = 4/15 + 64/75 = 84/75=28/25 (same as above).

It can be argued that by construction the portfolio created at time 0 would match the value of the derivative at time 1. Since, this value is the also the value of the portfolio required to be created at time 1 there is no leakage and hence the portfolio given above a self financing portfolio which replicates the value of the derivative.

e Use the same tree to price a European call option and use Black-scholes to price the European call option. BS can be taken as true and the difference between both of these gives the error adjustment in the above price.

European call option assuming the exercise price of 4 (at the money) from the above tree is (4/5)^2*(12)/4 = 48/25.

To calculate the black-scholes price we need to have the following parameters:

$r = \log_e(1+1/4)$ = 22.3144%

T = 2

S_0=K=4

Sigma is not given. However, we know that in the above case the up movement and down movement are reciprocal to each other u=1/d and we also know that a good estimator for

		u in such a case is $u = e^{\sigma\sqrt{t}}$. Hence, we can estimate sigma by $= \frac{1}{\sqrt{t}} log_e(u)$. In this case t=1 and hence sigma = log(2)=69.3147%. Substituting these values in the B/S equation we get call value = 2.067. The error is = 2.067 – 1.92 = 0.147. Therefore, using the control variate technique the price of the option in d would be approximated as = 28/25 + 2.067 – 1.92 = 28/25 + 0.147 = 1.267	
6	a	Portfolio effective duration is the weighted average of the effective durations of the portfolio bonds (500*100*8.5+250*94.3*1.75+300*93*8.5+100*9*4.5)/1119.75= 6.7 Change in portfolio value for change in one basis point =.01*6.7*1119.75 = 0.75 Lakh	3
	b	The 6.75% and 7% coupon bonds likely have no embedded options. For bonth of these bonds, modified duration and effective duration are identival, which would be the case if they had no embedded options	2
	c	The 8% bond is likely callable.It is trading at a premium, its effective duration is less than modified duration, and it exhibits negative convexity. 5.5% bond is likely putable, it is trading at a significant discount its effective duration is much lower than its modified duration and its convexity is positive but low.Note that a putable bond may be trade below par when put price is below par(also if there is risk that the issuer cannot honor the put). If it were callable we would expect its modified and effective durations to be closer in value because the market price is significantly below likely call price.	3
	d	Based on the effective duration and effective convexity of the 7% bond, the approximate price change is: =[-8.5*0.0025+125*00025^2]*300*.93=5.71Lakh	
	e	Effective duration is based on small changes in yield and is appropriate for parallel changed in the yield curve (or equal changes in the yields to maturity for all portfolio bonds). Other types of yield change will make portfolio duraction an inadequate measure of portfolio interest rate risk .	
7	a	Using of future instead of direct sales is having the following advantage a. Reduces the impact on the market price due to direct sales b. The commission charges for futures trading are relatively small as compared to other type of investments. c. Futures contracts are highly leveraged financial instruments which permit achieving greater gains using a limited amount of invested funds. d. It is possible to open short as well as long positions. Position can be	3

reversed easily.

e. Lead to high liquidity.

f. Spread the transaction cost over period of time

b

	Current	Future	current(Cr)	Future target(Cr)
Indian equity	50%	70%	50	70.00
UK equity	25%	15%	25	15.00
US equity	25%	15%	25	15.00

As we need to reduce the exposure in UK and US we need to have short position in the UK and the US futures and long position in Indian Equity futures.
The asset allocation should reflect the exposure at the date of when change is made, hence we need to short UK equity worth 10Cr and US equity worth 10 Cr. Future price is calculated by arbitrage free concept =S(0) * exp(rt)

	Amount(A)	Local currency ('000) (LC=A/FX)	Price of index	Int	Price of future (exp(r)* S(0)	Number of Future (LC/Future P)
India (BSE)	20	200000	18700	3%	19269	10380
UK equity	10	1111	5450	1%	5505	202
US equity	10	1818	13600	1%	13737	133

4

c Hedge for currency loss can be design by purchasing currency forwards.
We use interest rate party to determine the FX rate after 3 month. Use that to estimate amount of FX forward need to be purchased
Interest rate parity state that FX rate will change according to the differential of the interest rate in two economies

India interest 3%

Money received	Current	OS interest	New Fx=(exp(rindia)/exp(os)*startFX	OS leg	Indian Rupees
GBP	90	1%	91.82	5505*202	102104.76
USD	55	1%	56.11	13737*133	102514.15

4

d In-case of no hedge and same number of future index bought or sold

	Unit sold	Current price	New price(Old Price*Market fall)	Payment(Dom Curr)- Contract*Index price	Paym Rs(Fx paym
India	-10380	18700	15895	-164990.1	-1649
UK	202	5450	5177.5	1045.855	99356
US	133	13600	12512	1664.096	99845
					34211

Company Make profit of 34Lakh

In case of hedge

Expected Payment for future sold/bough t	Expected Payment for future sold/bought(local Curr)-Future priceX Number of future	Actual payment (Payment x currency)	In case of Currency Forward
Indian Equity	-200012.22	-200012.22	-200012.22
UK future	1112.01	105640.95	102104.76
US future	1827.02	109621.26	102514.15
Loss/Profit		15,249.99	4606.69

With Currency Hedge =4,6 Lakh

Without currency Hedge- 15.2 lakh

8

8 a CAP is an instrument which provides hedging against rise in interest rate. It consists of multiple options known as caplets which provide payment at each reset date if market specified interest rate is higher than the cap.

There will 8 options in 2 year cap with quarterly setting.

2

b We can construct the binomial tree by calculating risk neutral probability

Caplet payment

Max(0,Nominal(Rate-Strike))/(1+rate)

$$q = \frac{e^{r} - d}{u - d}$$

2

		Now we discount back the value of caplet to time t=1 by following formula Val(t=1)= exp(r(t=1) *(Vup*q + Vdown* (1-q)) Similarly we use value of node at year 1 to calculate value at year 0 r 8.64% v 83.76 R 7.20% V 79.09 r 6% v 67.27 r 6% v 51.31 R 4.80% V 33.53 r 3.84% v Similarly for second caplet r 7.20% v 51.31 r 6% v 35.66 R 4.80% V - Hence value of cap 67.27+35.66= 102.93 lakh	3 3
9	a	In an arbitrage-free market, risk-neutral valuation allows us to value an asset by discounting its cash flows at the risk-free rate and taking the average under the “risk-neutral measure”. In particular, consider a credit risky bond with maturity T . We assume that such a bond pays £1 if it	2

survives until maturity, or some fraction d (the recovery rate) if it has defaulted. By risk-neutral valuation the price of this bond can be written

$$v(0,T)=E_Q\left[e^{-\int_{t=0}^{T} r(t)dt}\left(I(\tau>T)+\delta I(\tau\le T)\right)\right]$$

Under assumption of constant interest rate this can be written as

2

$$v(0,T)=v^{GILT}(0,T)\left[1-\Pr_Q(\tau\le T)+\delta\Pr_Q(\tau\le T)\right]$$

Where vGilt(0,T) is price of risk free bond.

b We need to simulate the probability of default which is given by the random variable Y = $\varphi(-Z)$ = 1 - $\varphi(Z)$ = 1 - $\varphi\{\varphi^{-1}(U)\}$ =1-U where U is uniform (0,1) random variable and since $\varphi(Z)$ follows U(0,1) it follows that Z may be written as $\varphi^{-1}(U)$. Therefore, probability of default can be simulated by using uniform (0,1) as given above.

Now we can use 5 simulation with given random number to calculate the price of risky bond

Iteration	Random no	Prob of Default	Recovery 40%, Price=P(rf)*(1-DF+RE*DF)
1	0.985	1.5%	73.4151%
2	0.707	29.3%	61.0582%
3	0.447	55.3%	49.5015%
4	0.051	94.9%	31.8996%
5	0.783	21.7%	64.4364%
	average		56.0622%

The risky yield at a price of 56.0622% comes out to be $\frac{-\log_e 0.560622}{5}$= 11.57% compared to the risk free rate of 6% which implies the credit spread to be 5.57%.