Institute of Actuaries of India

Subject CT8 – Financial Economics

November 2012 Examinations

INDICATIVE SOLUTIONS

Solution 1 : (i) Consider the following two portfolios at time 0:

Portfolio A: A forward contract to buy one unit of the income generating asset at time T for K; simultaneously invest an amount Ke^{-rT} in a risk free asset.

Portfolio B: Buy e^{-qT} units of the asset, reinvesting the income generated as soon as it is received to buy more units of the asset.

Now consider the payoffs at time T:

Portfolio A: The risk free investment has grown to value K which is used to buy one unit of the asset using the forward contract.

Portfolio B: The amount of asset with the reinvested income has grown to one unit.

So, the two portfolios have same payoff at maturity.

By no arbitrage principle, the value of the two portfolios must be equal at time 0.

The cost of setting up Portfolio A is Ke^{-rT}.

The cost of setting up Portfolio B is S_0e^{-qT}.

$Ke^{-rT} = S_0e^{-qT} \Rightarrow K = S_0e^{(r-q)T}$.

[5]

(ii) From (i) we know that the zero-cost forward price for an asset generating continuous income is given by.

$K = S_0e^{(r-q)T}$

We note from the question that,

- USD is a continuous income generating asset at 5% per annum
- USD's current price is INR 53
- the risk free force of interest is 8% per annum
- the contract's revised time to maturity is 6 months.

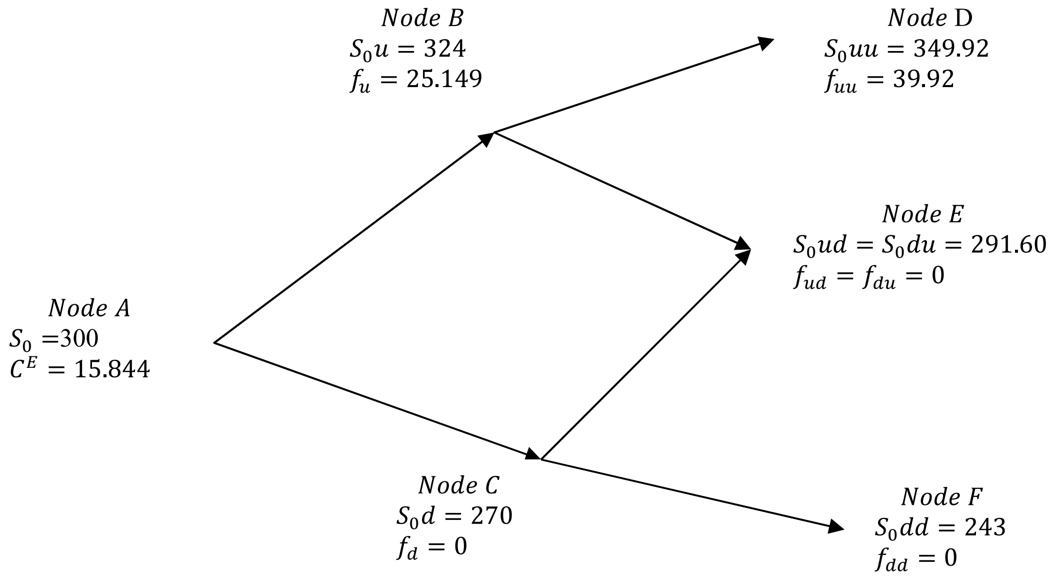
Also, Wipro currently owes INR 3 to SBI and this amount needs to be recovered through appropriately adjusting the forward price. Since SBI is buying the asset the forward price needs to be reduced with INR 3 rolled forward to maturity.

$$Therfore\ K = S_0 e^{(r-q)T} - 3e^{rT} = 53e^{(8\%-5\%)\frac{6}{12}} - 3e^{5\%*\frac{6}{12}} = 50.6786$$

[5]

[Total 10]

Solution 2 : **(i)** The binomial tree is as follows:



$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.06X0.25} - 0.90}{1.08 - 0.90} = 0.6395$$

$$f_u = e^{-0.06x0.25}[0.6395x39.92 + (1 - 0.6395)x0] = 25.1494$$
$$f_d = e^{-0.06x0.25}[0.6395x0 + (1 - 0.6395)x0] = 0$$
$$C^E = e^{-0.06x0.25}[0.6395x25.1494 + (1 - 0.6395)x0] = 15.8440$$

[4]

(ii) For an American put option

$$f_u^A = Max[f_u^E, 310 - 324] = 6.5341\ and\ f_d^A = Max[f_d^E, 310 - 270] = 40$$

$$P^A = e^{-0.06x0.25}[0.6395x6.5341 + (1 - 0.6395)x40] = 18.3211$$

Value of American Put Option = Rs. 18.3211

[3]

[Total 7]

Solution 3 : (i) Definitions as follows:

$c = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$,
where,
S_0 is share price at time 0
K *is the strike price*
r *is the risk free force of interest*
T *is time to maturity*

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

$\Phi(x)$is the cumulative distribution function of a standard normal variable
σ *is the volatility parameter*

[2]

(ii) $Ke^{-rT}\phi(d_2) = Ke^{-rT}\frac{1}{2\sqrt{\pi}}e^{-\frac{d_2^2}{2}}$

$$Now\ e^{-\frac{d_2^2}{2}} = e^{-\frac{(d_1-\sigma\sqrt{T})^2}{2}} = e^{-\frac{d_1^2-2d_1\sigma\sqrt{T}+\sigma^2T}{2}}$$

$$Ke^{-rT}\phi(d_2) = Ke^{-rT}e^{\frac{2d_1\sigma\sqrt{T}-\sigma^2T}{2}}\frac{1}{2\sqrt{\pi}}e^{-\frac{d_1^2}{2}}$$

$$= Ke^{-r(T-t)}e^{\frac{2\left(\log\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right)-\sigma^2T}{2}}\phi(d_1) = Ke^{-rT}\frac{S_0}{K}e^{rT}\phi(d_1) = S_0\phi(d_1)$$

Hence proved.

[4]

(iii) Differentiating the price of a call option with S_0

$$\frac{\partial c}{\partial S_0} = \frac{\partial}{\partial S_0}\left(S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)\right)$$
$$= \Phi(d_1) + S_0\phi(d_1)\frac{\partial d_1}{\partial S_0} - Ke^{-rT}\phi(d_2)\frac{\partial d_2}{\partial S_0}$$
$$Since\ \frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0}\ and$$
$$S_0\phi(d_1) = Ke^{-rT}\phi(d_2)$$
$$\frac{\partial c}{\partial S_0} = \Phi(d_1)$$

[4]

[Total 10]

Solution 4 : (i) If S_t denotes the market price of an investment, then the lognormal model of security prices states that, for $u > t$, log returns are given by:

$$\log(S_u) - \log(S_t) \sim N[\mu(u-t), \sigma^2(u-t)]$$

where μ is the drift, and σ is the volatility.

[2]

(ii) The continuous-time lognormal model may be inappropriate for behavior investment returns because:

- the volatility σ may not be constant over time. Estimates of volatility from past data are critically dependent on the time period chosen for the data and also how often the price history is sampled
- the long-term drift μ may not be constant over time. In particular, interest rates will impact the drift
- there is evidence in real markets of mean-reverting behavior, which is inconsistent with the independent increments assumption
- there is evidence in real markets of momentum effects, which is inconsistent with the independent increments assumption
- the distribution of security returns log (S_t/S_u) has a higher peak in reality than that implied by the normal distribution. This is because there are more days of little or no movement in the share price
- the distribution of security returns log (S_t/S_u) has fatter tails in reality than that implied by the normal distribution. This is because there are frequent big "jumps" in security prices.

[3]

(iii) A credit event is an event that will trigger the default of a bond and includes the following

- failure to pay either capital or a coupon
- loss event (ie where the company says that it is not going to make a payment)
- bankruptcy
- rating downgrade of the bond by a rating agency such as Standard and Poor's or Moody's

[2]

[Total 7]

Solution 5 : **(i)** The equations for short rate under the three models are:

$$Vasicek\ model: dr(t) = \alpha[\mu - r(t)]dt + \sigma dW_t$$

$$CIR\ model: dr(t) = \alpha[\mu - r(t)]dt + \sigma\sqrt{r(t)}dW_t$$

$$HW\ model: dr(t) = \alpha[\mu(t) - r(t)]dt + \sigma dW_t$$

Since the coefficient of random components under Vasicek and HW model do not depend on short rate, *r(t)*, any change in the short rate will not impact the volatility in short-rate.

However, the volatility under CIR model increases in line with square root of r(t). So, as short-rate rises, the underlying volatility also increases.

[4]

(ii) $For\ brevity\ lets\ denote\ the\ Vasicek\ model\ as$

$$dr = \alpha[\mu - r]dt + \sigma dW$$
$$dr + \alpha r dt = \alpha\mu dt + \sigma dW$$

Multiply both sides by the integrating factor $e^{\alpha t}$

$$e^{\alpha t}dr + \alpha r e^{\alpha t}dt = \alpha\mu e^{\alpha t}dt + \sigma e^{\alpha t}dW$$
$$d(e^{\alpha t}r) = \alpha\mu e^{\alpha t}dt + \sigma e^{\alpha t}dW$$

Integrating both sides from 0 to t

$$\int_0^t d(e^{\alpha s}r) = \int_0^t \alpha\mu e^{\alpha s}ds + \int_0^t \sigma e^{\alpha s}dW$$

$$e^{\alpha t}r_t - r_0 = \int_0^t \alpha\mu e^{\alpha s}ds + \int_0^t \sigma e^{\alpha s}dW$$

$$r(t) = r(0)e^{-\alpha t} + \int_0^t \alpha\mu e^{(s-t)\alpha}ds + \int_0^t \sigma e^{(s-t)\alpha}dW$$

[5]

[Total 9]

Solution 6 : (i) The cumulative distribution function

$$F(x) = \int_0^x cky^{c-1}(1+y^c)^{-(k+1)}dy$$

$$Let\ y^c = z; cy^{(c-1)}dy = dz; for\ y = 0, z = 0; for\ y = x, z = x^c$$

$$\int_0^{x^c} k(1+z)^{-(k+1)}dz = \left[\frac{-1}{(1+z)^k}\right]_0^{x^c} = 1 - \frac{1}{(1+x^c)^k}$$

[3]

Consider two investment portfolios, A and B. Let F_A and F_B be the cumulative distribution function of return provided by A and B respectively.

(ii) (a) Absolute dominance: Absolute dominance exists when one investment portfolio provides a higher return than another in all possible circumstances. This situation rarely occurs.

(b) First-order stochastic dominance: For an investor preferring more to less, A has first-order stochastic dominance over B, if:

$$F_A(x) \leq F_B(x), for\ all\ x, and$$
$$F_A(x) < F_B(x), for\ some\ value\ of\ x$$

(c) Second-order stochastic dominance: For a risk averse investor preferring more to less, A has second-order stochastic dominance over B if:

$$\int_a^x F_A(y)dy \leq \int_a^x F_B(y)dy\ , for\ all\ x, and$$
$$\int_a^x F_A(y)dy < \int_a^x F_B(y)dy\ , for\ some\ value\ of\ x, where$$
$$a\ is\ the\ lowest\ return\ that\ the\ portfolios\ can\ possibly\ provide.$$

[4]

(iii) We should ascertain if either portfolio has first order stochastic dominance over another.

$$F_A(x) = 1 - \frac{1}{(1+x^1)^1}\ and\ F_B(x) = 1 - \frac{1}{(1+x^1)^2}$$

Consider the mathematical difference of the two cumulative distribution functions.

$$F_A(x) - F_B(x) = \frac{1}{(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{(1+x) - (1+x)^2}{(1+x)^3} = \frac{(1+x)(1-1-x)}{(1+x)^3}$$
$$= -\frac{x}{(1+x)^2} < 0;\ for\ all\ x > 0$$

This proves that A has first-order stochastic dominance over B.

An investor who prefers more to less should choose A over B.

[4]

(iv) Similarly, consider the following:

$$F_A(x) - F_C(x) = \frac{1}{1+x^2} - \frac{1}{1+x}$$

$$= \frac{(1+x) - (1+x^2)}{(1+x)(1+x^2)}$$
$$= \frac{x(x-1)}{(1+x)(1+x^2)} \begin{cases} > 0\ for\ x < 1 \\ 0\ for\ x = 1 \\ < 0\ for\ x > 1 \end{cases}$$

Hence, neither portfolio has first order stochastic dominance over another.

[5]

[Total 16]

Solution 7 : (i) Expected shortfall below a certain level 'L' is given by:

$$Expected\ shortfall\ =\ E[max(L - X, 0)]\ =\ \int_{-\infty}^{L} (L - X) f(x) dx$$

[1]

(ii) $Shortfall\ Probability\ =\ P[R < L]$

$$P[R < 0\%] =\ P[1\%X - 0.5\%(12 - X) < 0\%]$$
$$=\ P[1.5\%X - 6\% < 0\%] = P[X < 4]\ =\ P[Bin(12{,}0.5) < 4]$$
$$P[Bin(12{,}0.5) = k] = \frac{12!}{k!(12-k)!} 0.5^k (1 - 0.5)^{12-k}$$

The table below provides the necessary numbers for remaining parts of the question.

X	*R*	*p*	*Cumulative*	*Shortfall (L-R)*	*(L-R)*p*	$(L-r)^2 * p$
0	-6%	0.0002	0.02%	6%	0.0015%	0.000088%
1	-4.5%	0.0029	0.32%	4.5%	0.0132%	0.000593%
2	-3%	0.0161	1.93%	3%	0.0483%	0.001450%
3	-1.5%	0.0537	7.30%	1.5%	0.0806%	0.001208%

4	0%	0.1208	19.38%			
				Total	0.1436%	0.00334%

$P[R < 0\%] = 7.30\%$

For discrete variables, the expected shortfall can be calculated as
$\sum_{\mathrm{R_{min}}}^{L} (\mathrm{L} - \mathrm{r})\mathrm{P}(\mathrm{R} = \mathrm{r}) = 0.1436\%$

[7]

(iii) Shortfall variance = 0.00334%
(The calculations are reflected in the table above)

[4]

(iv) The main weakness of Value-at-Risk (VaR) as a measure for risk is that it does not quantify the size of the "tail".

Expected shortfall on the other hand measures the riskiness of a portfolio by considering both the size and the likelihood of losses versus a benchmark.

This move by the Basel Committee will therefore account for tail risk in a more comprehensive manner.

[4]

[Total 16]

Solution 8 : (i) We can use the following equations:

- $E[R_i] = E[a_i] + b_{i,1}E[I_1] + b_{i,2}E[I_2] + b_{i,3}E[I_3]$

We know that $Cov(I_k, I_l) = 0$ since they are orthogonal.
We assume that $Cov(a_i, a_j) = 0 \; \forall \; i \neq j and \; Cov(a_i, I_k) = 0 \; \forall \; i, k.$

- $Var[R_i] = Var[a_i] + {b_{i,1}}^2 Var[I_1] + {b_{i,2}}^2 Var[I_2] + {b_{i,3}}^2 Var[I_3]$
- $Cov(R_i, R_j) = Cov(a_i + b_{i,1}I_1 + b_{i,2}I_2 + b_{i,3}I_3, a_j + b_{j,1}I_1 + b_{j,2}I_2 + b_{j,3}I_3)$

$$= \sum_{k=1}^{3} b_{i,k} b_{j,k} \, Var(I_k)$$

$E[R_X] = 6\% + 1 * 2\% = 8\%$
$E[R_Y] = 5\% + 0.5 * [2\% + 3\% + 4\%] = 9.5\%$
$E[R_Z] = 4\% + 1 * 3\% + 0.5 * 4\% = 9\%$

$Var[R_X] = 9\%\% + 1^2 * 1\%^2 = 0.1\% \; or \; 10\%\%$
$Var[R_Y] = 4\%\% + 0.5^2 * [1\%^2 + 2\%^2 + 4\%^2] = 0.0925\% \; or \; 9.25\%\%$
$Var[R_Z] = 1\%\% + 1^2 * 2\%^2 + 0.5^2 * 4\%^2 = 0.09\% \; or \; 9\%\%$

$Cov(R_X, R_Y) = 1 * 0.5 * 1\%^2 = 0.005\% \; or \; 0.5\%\%$
$Cov(R_Y, R_Z) = 1 * 0.5 * 4\%^2 + 0.5 * 0.5 * 4\%^2 = 0.06\% \; or \; 6\%\%$
$Cov(R_X, R_Z) = 0\%$

[7]

(ii) The Lagrangian function is:

$$W = V - \lambda(E - E_P) - \mu(\textstyle\sum_i x_i - 1)$$

where:

$V \; is \; the \; portfolio \; variance$
$E \; is \; the \; expected \; return \; on \; the \; portfolio$
$x_i is \; the \; proportion \; of \; portfolio \; invested \; in \; security \; i$
$E_P \; and \; 1 \; are \; constraining \; constants \; and$
$\lambda \; and \; \mu \; are \; Lagrangian \; multipliers$

[3]

(iii) To solve the minimization problem we set the partial derivatives of W with respect to all the $x_i \; and \; \lambda \; and \; \mu$ equal to zero.

$$W = \textstyle\sum_i \sum_j C_{ij} x_i x_j - \lambda(\sum_i E_i x_i - E_P) - \mu(\sum_i x_i - 1)$$

$$\frac{\partial W}{\partial x_i} = 2\sum_j C_{ij} x_j - \lambda E_i \; - \mu = 0|_{i=1,2,3}$$

$$\frac{\partial W}{\partial \lambda} = -\left(\sum_i E_i x_i - E_P\right) = 0$$

$$\frac{\partial W}{\partial \mu} = -\left(\sum_i x_i - 1\right) = 0$$

These five equations can be represented in matrix form as follows:
$Ay = b$

$$A = \begin{bmatrix} 2C_{11} & 2C_{12} & 2C_{13} & -E_1 & -1 \\ 2C_{21} & 2C_{22} & 2C_{23} & -E_2 & -1 \\ 2C_{31} & 2C_{32} & 2C_{33} & -E_3 & -1 \\ E_1 & E_2 & E_3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}; y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda \\ \mu \end{bmatrix}; b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_P \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.2\% & 0.01\% & 0\% & -8\% & -1 \\ 0.01\% & 0.185\% & 0.12\% & -9.5\% & -1 \\ 0\% & 0.12\% & 0.18\% & -9\% & -1 \\ 8\% & 9.5\% & 9\% & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

[7 marks]

(iv) $Ay = b \Rightarrow y = A^{-1}b$

The first three items of $A^{-1}b$ are $x_1, x_2 and\ x_3$ respectively.

$$A^{-1} = \begin{bmatrix} 86.21 & 172.41 & -258.62 & -55.17 & 5.22 \\ 172.41 & 344.83 & -517.24 & 89.66 & -7.55 \\ -258.62 & -517.24 & 775.86 & -34.48 & 3.33 \\ 55.170 & -89.66 & 34.48 & 14.69 & -1.28 \\ -5.22 & 7.55 & -3.33 & -1.28 & 0.11 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_P \\ 1 \end{bmatrix}$$

$$x_1 = -55.17E_P + 5.22$$
$$x_2 = 89.66E_P - 7.55$$
$$x_3 = -34.48E_P + 3.33$$

[4]

[Total 21]

Solution 9 : (i) A market is said to be semi-strong form efficient if the price of each security in such market reflects all publicly available information. **[1]**

(ii) Possible explanations for such proofs:

- many published tests make implicit, but possibly invalid, assumptions for example, normality of return, or stationarity of time series
- some of the differences are purely differences of terminology, for example, do we regard anomalies as disproving EMH, if transaction costs prevent their exploitation?
- the tests may not have made appropriate allowance for risk – the EMH is not contradicted by a strategy that produces higher profits than the market portfolio by taking higher risks.

[3]

[Total 4]