Institute of Actuaries of India

Subject CT1 – Financial Mathematics

November 2012 Examinations

INDICATIVE SOLUTIONS

Note

1. **Markers are instructed to follow the guidelines as closely as possible while giving step marks**
2. **No step marks need to be given for answers lacking fundamental understanding of the concepts**
3. **Any alternative solution should be given full credit provided it is logical**
4. **Any incorrect solution should not be penalized more than once**

Sol. 1)

$$A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)dt\right) = \exp\left(\int_{t_1}^{t_2} 0.08 + \frac{b}{t+1}dt\right) \quad (0.5)$$

$$= \exp\left(0.08t\right)_{t_1}^{t_2} * \exp\left(b * \ln(t+1)_{t_1}^{t_2}\right)$$

$$= \exp\left(0.08 * (t_2 - t_1)\right) * \exp\left(b * \ln\left(\frac{t_2+1}{t_1+1}\right)\right)$$

$$= \exp\left(0.08 * (t_2 - t_1)\right) * \left(\frac{t_2+1}{t_1+1}\right)^b \quad \text{.....................................}(1) \quad (1)$$

Using $t_1 = 0$ and $t_2 = 5$ we get

$1,546.25 = 1,000 * (\exp(0.4) * 6^b)$ (0.5)

$=> 1,546.25 = 1,000 * (1.49182 * 6^b)$

$=> 1.03648 = 6^b$

$=> \ln(1.03648) = b \ln 6$

$=> b = 0.02$ (1.5)

The amount X that should be invested at time $t_1 = 2$ to obtain accumulated amount of ` *1,000* at time $t_2 = 8$ is given by the equation

$1,000 = X * (\exp(0.48) * 3^{0.02})$ using equation *(1)* (0.5)

$=> X = 1,000 / (1.6161 * 1.02222)$

$=> X = 605.33$ (1)

[5]

Sol. 2)

(i)

Money weighted rate of return i should satisfy following equation of value

$$95,000 * (1+i)^6 + 9,000 * (1+i)^3 = 1,65,065$$

$$=> 95,000 * (1+i)^6 + 9,000 * (1+i)^3 - 1,65,065 = 0$$ (0.5)

The roots of above quadratic equation are given by

$$(1+i)^3 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ where $a = 95,000$, $b = 9,000$, $c = -1,65,065$

$$=> (1+i)^3 = \frac{-9,000 \pm \sqrt{9,000^2 + 4 * 95,000 * 1,65,065}}{2 * 95,000}$$

$=> (1+i)^3 = 1.27164$ (taking the positive root)

$=> i = 8.34\%$

(2)

Hence TWRR = MWRR / 1.25 = 8.34% / 1.25 = 6.672%

(0.5)

The fund value X, as on 30th June 2009 should satisfy

$$(X / 95,000) * (1,65,065 / (9,000 + X)) = (1 + 6.672\%)^6$$

$$X / (9,000 + X) = 0.84795$$ (1)

X = ₹50,191.06

(1)

Alternate Solution: **[5]**

Money weighted rate of return i should satisfy following equation of value

$$95,000 * (1+i)^6 + 9,000 * (1+i)^3 = 1,65,065$$

By trial and error (0.5)

if $i = 8\%$, LHS = 1,62,090.47

if $i = 8.5\%$, LHS = 1,66,485.02

Hence by interpolation we get MWRR

$i = 8\% + (8.5\% - 8\%) * (1,65,065 - 1,62,090.47) / (1,66,485.02 - 1,62,090.47)$

$\approx 8.34\%$ (2)

Hence TWRR = MWRR / *1.25 = 8.34% / 1.25 = 6.672%* (0.5)

The fund value *X*, as on 30th June 2009 should satisfy

$(X / 95,000) * (1,65,065 / (9,000 + X)) = (1 + 6.672\%)^6$ (1)

$X / (9,000 + X) = 0.84795$

$X =$ ₹*50,191.06* (1)

(5)

(ii)

Time Weighted Rate of Return (TWRR) is better measure of fund manager's performance. (0.5)

Money-weighted rate of return (MWRR) is sensitive to the amounts and timings of the net cash flows. But fund manager does not have any control on these.

Time-weighted rate of return (TWRR) is calculated using the "growth factors" reflecting the change in fund value between the times of consecutive cash flows. Thus TWRR is independent of amount and timing of cash-flows.

Hence Time-weighted rate of return (TWRR) is better measure of fund manager's performance.

(1.5)

(2)

[7]

Sol. 3) $i^{(4)} = 4((1+i)^{0.25} - 1) = 5.39007\%$ (0.5)

g(1-income tax) = *7.5% (1 – 25%) = 5.625%* (0.5)

Since $i^{(4)}$ < g(1-income tax), there is a capital loss; Hence if the borrower redeems at the earliest possible date he would be paying maximum price. (1)

Hence the required price need to be calculated assuming the bond will be redeemed at the earliest possible date i.e. term of 8 years. (0.5)

Thus the required price per unit of bond, P

$= 7.5 * (1 - 25\%) * a_{\overline{8}|}^{(4)} + 100 * v^8$ *@5.5% p.a. effective.* (1.5)

Where $a_{\overline{8}|}^{(4)} = \dfrac{1-(1+i)^{-8}}{i^{(4)}} = 6.46376$ (1)

$= 5.625 * 6.46376 + 100 * 1.055^{-8} = ₹101.52$ (1)

[6]

Sol. 4)

(i)

(a)

$i = 7.5\%$

$$\ddot{s}_{\overline{8}|} = \frac{(1+i)^8 - 1}{d} = (1+i)\frac{(1+i)^8 - 1}{i}$$ (1)

$= 11.22985$ (1)

(2)

(b)

$$\delta = \ln(1+i) = 7.23207\%$$ (1)

$${}_{2|}\bar{a}_{\overline{5}|} = v^2 \frac{1 - v^5}{\delta}$$ (1)

$= 3.63074$ (1)

(3)

(c)

$$i^{(4)} = 4((1+i)^{1/4} - 1) = 7.29784\%$$ (1)

$$a_{\overline{\infty}|}^{(4)} = \frac{1}{i^{(4)}} = 13.70268$$ (1)

(2)

(ii)

$$i = (1 + (i^{(2)}/2))^2 - 1 = 10.25\%$$ (0.5)

$$d = \frac{i}{1+i} = 0.0929705$$ (0.5)

The accumulated value of investment made continuously starting from time t = 0,

$$10{,}000 * \bar{s}_{\overline{3}|} = 10{,}000 * \frac{(1+i)^3 - 1}{\delta} = 10{,}000 * \frac{(1+i)^3 - 1}{\ln(1+i)}$$ (1)

= 10,000 * 3.48529 = `34,852.9 (1)

The accumulated value of investment made at the beginning of 2nd year,

$$10{,}000 * \ddot{s}_{\overline{2}|}^{(12)} = 10{,}000 * \frac{(1+i)^2 - 1}{d^{(12)}}$$ (1)

where $d^{(12)} = 12*(1-(1-d)^{1/12})$ = 0.0971846 (0.5)

= 10,000 * 2.21749 = `22,174.9 (1)

The accumulated value of investment made at the beginning of 3rd year,

$10{,}000 * \ddot{s}_{\overline{1}|}^{(2)} = 5{,}000 * (1+i^{(2)}/2))^2 + 5{,}000 * (1 + i^{(2)}/2)$ (1)

$= 5{,}000 * 1.05^2 + 5{,}000 * 1.05 =$ `10,762.5 (1)

The accumulated value of all the investments = 34,852.9 + 22,174.9 + 10,762.5

= ₹67,790.3

(0.5)

(8)

[15]

Sol. 5)

(i)

a) Characteristics of Eurobond:

Interest Bearing Securities where regular interest is paid with final capital redemption at par.

Issued by large companies, governments and supra-national organizations.

Issued and traded Internationally and often not denominated in a currency native to the country of the issuer.

Form of unsecured medium or long term borrowing.

Yield depends upon owner and issue size but typically slightly lower than for conventional unsecured bonds of the same issuer.

Usually issued in bearer form.

(3)

b) Characteristics of Short interest rate future:

Based on benchmark interest rate and settled for cash. No principal or interest changes hands.

The contract is based on an interest paid on a notional deposit for a specified period from the expiry of the future.

The contract is settled for cash

On expiry the purchaser will make profit (loss) related to difference between purchase price and settlement price.

The party delivering the contract will have made corresponding loss (profit).

The quotation is structured in a way such that if interest rate rises the price of the future falls and vice versa.

(One mark for each distinct relevant point up to a maximum of 3)

(3)

(6)

(ii) $y_1 = 7\%, \quad f_{1,4} = 7.5\%, \quad f_{5,5} = 9\%$

$$(1+y_5)^5 = (1+y_1)(1+f_{1,4})^4$$

$$= 1.07 * (1.075)^4 = 1.42895$$ [1]

$$y_5 = (1.42895)^{(1/5)} - 1 = 7.4\%$$ [0.5]

$$(1+y_{10})^{10} = (1+y_1)(1+f_{1,4})^4(1+f_{5,5})^5$$

$$= 1.07 * (1.075)^4 * (1.09)^5 = 2.19862$$ [1]

$$y_{10} = (2.19862)^{(1/10)} - 1 = 8.19695\%$$ [0.5]

The values of 1 year, 5 year and 10 year spot yields show that it is an upward sloping yield curve. According to expectation theory of interest rate a yield curve can be upward sloping if the investors expect short term rates to increase in future.

(1)

(4)

(iii)

$\delta = 7\%, \quad v = exp(-\delta) = 0.93239$ (0.5)

Present value of dividends payable till the end of 5^{th} year

$= 1*1.02\,v + 1*1.02*1.04 * v^2\,(1 + 1.06\,v + 1.06^2 * v^2 + 1.06^3 * v^3)$ (2)

= `4.57581 (1)

Forward value of the contract at the end of 5^{th} year

$= (75 - 4.57581) * exp(5\delta) = 99.93668 \approx$ ₹100 (1.5)

(5)

[15]

Sol. 6)

a) $Exp(\mu + 0.5*\sigma^2) = 1.06$

$Exp(2\mu + \sigma^2)(Exp(\sigma^2)-1) = (0.02)^2$

Hence $Exp(\sigma^2)-1 = (0.02)^2/(1.06)^2$

$\sigma^2 = 0.00035594, \quad \sigma = 0.01897, \quad \mu = 0.05809$ (2)

Accumulated amount A at t = 4 for an investment of 1 at t= 0 has following distribution

$Log\,A \sim N(4\mu, 4\sigma^2)$ (1)

Hence for accumulated amount A

Mean = $Exp(4\mu + 0.5*4*\sigma^2) = 1.26248$ (1)

Variance = $Exp(8\mu + 4*\sigma^2)(Exp(4\sigma^2)-1) = 0.002296$

Standard deviation = 0.04792 (1)

(5)

Alternate Solution

Let j and s be the mean and standard deviation of i.

Hence, $E(1+i) = 1 + j = 1.06$

and $Var(1 + i) = Var(i) = 0.02^2 = 0.004$ (0.5)

(0.5)

For accumulated amount the mean is given by

$$E[S_4] = \prod_{t=1}^{4} E(1+i_t) = (1+j)^4$$ (1)

$$= 1.06^4$$

$$= 1.26248$$ (0.5)

The variance of accumulated amount

$$Var[S_4] = ((1+j)^2 + s^2)^4 - (1+j)^{2*4}$$ (1)

$$= (1.06^2 + 0.02^2)^4 - 1.06^8$$

$$= 0.002271$$ (1)

Standard deviation of accumulated amount $=0.04765$

(0.5)

(5)

b) The accumulated amount at the end of fifth year A_5 should satisfy

$$A_5 = (15{,}000 * (1+i_1) + 12{,}000) * A_4$$

(0.5)

Hence Mean $E(A_5) = (15{,}000 * E(1+i_1) + 12{,}000) * E(A_4)$

$$= (15{,}000 * 1.06 + 12{,}000) * 1.26248$$

$$= ₹35223.19$$ (1)

$$E[(A_5)^2] = (15{,}000^2 * E[(1+i_1)^2] + 2 * 15000 * E(1+i_1) * 12{,}000 + 12{,}000^2) * E[(A_4)^2]$$

$$= (15{,}000^2 * (1.06^2 + 0.02^2) + 2 * 15000 * 1.06 * 12{,}000 + 12{,}000^2) * (1.26248^2 + 0.002296) \text{ since } E[(A_4)^2] = V(A_4) + [E(A_4)]^2$$ (2)

$$= ₹\ 1{,}24{,}26{,}04{,}138$$ (1)

Variance $VAR(A_5) = E[(A_5)^2] - [E(A_5)]^2$

$$= 1{,}24{,}26{,}04{,}138 - 35{,}223.19^2$$

$$= 19{,}31{,}024.22$$ (1)

Standard deviation of A_5 = ₹ 1,389.61

(0.5)

(6)

[11]

Sol. 7) The net present value of the cash-flow at 14% per annum effective is given by (in ` Crores)

$NPV = -7.5 - 5v - 2.5v^2 + v^{3.5}[3 + (3 + 0.5)v + (3 + 2 * 0.5)v^2 + ... + (3 + 6 * 0.5)v^6]$
$- 0.25v^4(1 + 1.05v + ... + 1.05^4 v^4) + 17v^{10}$ @ 14% per annum effective (1.5)

$NPV = -13.80963 + \{3 * v^{3.5} * \ddot{a}_{\overline{7}|} + 0.5 * v^{3.5} * I\ddot{a}_{\overline{6}|}\} - 0.25v^4 \ddot{a}_{\overline{5}|}^{@j} + 4.58564$ (1.5)

where $j = 1.14/1.05 - 1 = 8.57143\%$ (1)
$v = (1+i)^{-1} = 0.87719$

$$\ddot{a}_{\overline{7}|} = \frac{1-v^7}{i/(1+i)} = 4.88867$$ (0.5)

$$\ddot{a}_{\overline{6}|} = \ddot{a}_{\overline{7}|} - v^6 = 4.43308$$ (0.5)

$$Ia_{\overline{6}|} = \frac{\ddot{a}_{\overline{6}|} - 6v^6}{i} = 12.13973$$ (1)

$$\ddot{a}_{\overline{5}|}^{@j} = \frac{1-(1+j)^{-5}}{j/(1+j)} = 4.27043$$ (1)

Hence,

$NPV = -13.80963 + \{3 * v^{3.5} * 4.88874 + 0.5 * v^{3.5} * 12.14069\} - 0.25v^4 * 4.27043 + 4.58564$ (0.5)

$NPV = -13.80963 + 13.10857 - 0.63211 + 4.58564 =$ ₹ 3.25247 Crores (0.5)

Hence the project is viable at 14% per annum effective rate of return. The film producer should invest into the project. (1)

[9]

Sol. 8)
(i)

$$i^{(12)} = 12((1+i)^{(1/12)} - 1) = 9.56897\%$$ (0.5)

$$a_{\overline{5}|}^{(12)} = \frac{1-(1+i)^{-5}}{i^{(12)}} = 3.96154$$ (0.5)

Loan Amount, *5,00,000 =12 * EMI ** $a^{(12)}_{\overline{5}|}$

*= 12 * EMI * 3.96154* (0.5)

EMI =₹ 10,517.795 (0.5)

(2)

(ii)

(a)

Loan Outstanding at end of 12 months *= 12 * EMI ** $a^{(12)}_{\overline{4}|}$

$$a^{(12)}_{\overline{4}|} = \frac{1-(1+i)^{-4}}{i^{(12)}} = 3.31265$$ (0.5)

Loan Outstanding at end of 12 months = *12 * 10,517.795 * 3.31265*

= 4,18,101.28 (1)

Revised interest rate = *12%*

Revised interest rate per month = *0.94888%* (0.5)

The number of months n, over which payment need to be made is given by the equation,

*4,18,101.28 = 10,517.795 ** $a_{\overline{n}|}$ at the rate of interest *0.94888%* (1)

$$\frac{1-(1+i)^{-n}}{i} = 39.7518$$

n = 50.13994 ≈ 51 months (1)

(4)

(b) The last payment will be made at the end of 51st month.

Hence, the last payment R should satisfy following equation of value

*4,18,101.28 = 10,517.795 ** $a_{\overline{50}|}$ *+ R* v^{51} at the rate of interest *0.94888%*

*= 10,517.795 * 39.665 + 0.61776 * R* (1.5)

Hence the amount of last payment R= *1,477.83* (0.5)

(2)

(iii)

Accumulated value of outstanding loan at the end of 12 months from now @ 12 % (new i) per annum = *4,18,101.28 * (1 + 12%) = 4,68,273.43* (1)

$i^{(12)} = 12((1+i)^{(1/12)} - 1) = 11.38655\%$ (0.5)

$a_{\overline{3}|}^{(12)} = \frac{1-(1+i)^{-3}}{i^{(12)}} = 2.53123$ (0.5)

The required equal monthly installment EMI_{Rev} is given by

$4,68,273.43 = 12 * EMI_{Rev} * a_{\overline{3}|}^{(12)}$ at *12%* per annum effective

$= 12 * EMI_{Rev} * 2.53123$

EMI_{Rev} =₹ *15,416.53* (1)

(3)

[11]

Sol.9)

Total amount invested = ₹ *6,000*

Amount invested in equity *= 25% * 6,000 =* ₹ *1,500*

Amount invested in bond = ₹ *4,500* (0.5)

Number of equity share units purchased for the investor = *1,500 / 12 = 125* (1)

The IRR (i) should satisfy following equation of value

$125 * (0.35\,v + 0.4\,v^2 + 0.475\,v^3 + 0.5\,v^4 + 0.45\,v^6 + 37\,v^6)$

$+ 4,500\,(7.5\%\ a_{\overline{6}|} + 105\%\ v^6) = 6,000$ where $v = (1+i)^{-1}$ (3)

Return on equity without dividend is $(37/12)^{(1/6)} - 1 = 20.6\%$

Return on bond will be higher than *7.5%.*

Hence average return on portfolio will be higher than

*25% * 20.6% + 75% * 7.5% = 10.78%*

By trial and error

if $i = 12\%$ $v = 0.89286$

$a_{\overline{6}|} = 4.11133$

$LHS = 125 * 20.26094 + 4{,}500 * 0.84032 = 6{,}314.0575$ (1.5)

if $i = 13\%$ $v = 0.88496$

$a_{\overline{6}|} = 3.99744$

$LHS = 125 * 19.24731 + 4{,}500 * 0.80416 = 6{,}024.63375$ (1.5)

Hence by interpolation we get MWRR

$i = 12\% + (13\% - 12\%) * (6{,}000 - 6{,}314.0575) / (6{,}024.63375 - 6{,}314.0575)$

$= 13.09$

$\approx 13\%$ correct up to one integer value (1)

(0.5)

[9]

Sol. 10)

(i)

According to Redington's immunisation theory a portfolio of assets and liabilities can be immunized against small change of interest rate if following three conditions are satisfied:

1. Present value of Asset cash-flows = Present value of Liability cash-flows at starting rate of interest (0.5)

2. Volatility of Asset cash-flows = Volatility of Liability cash-flows (0.5)

3. Convexity of Asset cash-flows > Convexity of Liability cash-flows. (1)

(2)

(ii)

Interest rate $i = 7\%$, $v = 0.93458$

Present value of liability, $PV_L = 7\,v^6 + 1\,v^9 + 2\,v^{11}$ (1)

$= ₹\ 6.15854$ Lakh (1)

Volatility of liability, Vol_L,

$$-PV'_L / PV_L = (7 * 6 * v^7 + 1 * 9 * v^{10} + 2 * 11 * v^{12}) / PV_L \quad (1)$$

$$= 6.57609 \quad (1)$$

(4)

(iii)

$$a_{\overline{5}|} = \frac{1 - v^5}{i} = 4.10017 \text{ @ } 7\% \text{ p.a.} \quad (0.5)$$

Present value of asset (in ` Lakhs),

$$PV_A = 0.15 * a_{\overline{5}|} + 3\,v^5 + R\,v^n$$

$$= 0.15 * 4.10017 + 3 * 0.71299 + R\,v^n$$

$$= 2.754 + R\,v^n \text{ in ` Lakh} \quad (1)$$

Volatility of asset, Vol_A, $-PV'_A / PV_A = v\,(0.15 * Ia_{\overline{5}|} + 3 * 5\,v^5 + nR\,v^n) / PV_A$ (0.5)

$$\ddot{a}_{\overline{5}|} = (1+i) * a_{\overline{5}|} = 4.38718 \quad (0.5)$$

$$Ia_{\overline{5}|} = \frac{\ddot{a}_{\overline{5}|} - 5v^5}{i} = 11.74626 \quad (0.5)$$

Hence $Vol_A = v * (12.45676 + nR\,v^n) / PV_A$

To satisfy first condition of Redington's theory

$2.754 + R\,v^n = 6.15854$

$\Rightarrow R\,v^n = 3.40454$ *@ 7% p.a.* --------------------- *{Equation A}* (1)

To satisfy second condition of Redington's theory

$v * (12.45676 + nR\,v^n) / PV_A = 6.57609$ (0.5)

Using Equation A this becomes,

$\Rightarrow 12.45676 + 3.40454\,n = 6.57609 * 6.15854 * (1 / v)$ since $PV_A = PV_L$ (0.5)

$\Rightarrow n = 9.06$ (0.5)

=> *R* = ₹ *6.28453* Lakhs (0.5)

(6)

[12]

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