Institute of Actuaries of India

Subject ST6 – Finance and Investment B

November 2011 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1. When the asset price is positively correlated with volatility, the volatility tends to increase as the asset price increases, producing the thin left tail and fat right tail. Implied volatility then increases with the strike price.

[2]

2. (a) Suppose V_1 is the expected value of the business entity at the end of one year and D is the face value of the debt. The value of the Ramesh's's position in one year is:

$\text{Max}(V_1 - D, 0)$

This is the payoff from a call option on value of the business entity with strike price D and time to maturity 1 year.

(b) The debt holders get $\text{Min}(V_1, D) = V_1 - \text{Max}(V_1 - D, 0)$
This is a long position in the assets of the business entity combined with a short position in a call option on value of the business entity with strike price D and time to maturity 1 year.

(c) The debt holders get $\text{Min}(V_1, D) = D - \text{Max}(D - V_1, 0)$

The payoff is similar to that the debt holders have made a risk-free loan (worth D after 1 year) combined with a short position in a put option on the value of the business entity with strike price D and time to maturity 1 year.

(d) Ramesh can increase the value of his position by increasing the value of the call option in (a). It follows that Ramesh should attempt to increase both V and volatility of V. Increasing the volatility of V is beneficial to Ramesh. If there is a large increase in V, Ramesh benefits to the full extent of the change. If there is a large decrease in V, much of the downside risk is absorbed by the debt holders of the business entity.

[8]

3. (a) A set of probability ratios $\frac{q_i}{p_i}$ is called a Radon – Nikodym derivative of Q with respect to P and is written as $\frac{dQ}{dP}$ so that $E_Q[C] = E_P\left[\frac{dQ}{dP}C\right]$.
Radon – Nikodym derivative applies if and only if P and Q are equivalent probability measures.

(b) The Radon – Nikodym derivative $\frac{dQ}{dP}$ is given by:

$$\frac{dQ}{dP} = \frac{\frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\tilde{W}_T^2 / T}}{\frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}W_T^2 / T}}$$

which implies

$$\frac{dQ}{dP} = e^{-\frac{1}{2T}\left(\tilde{W}_T^2 - W_T^2\right)}$$

$$\frac{dQ}{dP} = e^{-\frac{1}{2T}\left((W_T+\mu T)^2 - W_T{}^2\right)}$$

$$\frac{dQ}{dP} = e^{-\frac{1}{2T}\left(2\mu T W_T + (\mu T)^2\right)}$$

$$\frac{dQ}{dP} = e^{-\mu W_T - \frac{1}{2}\mu^2 T}$$

(c) The moment generating function of $\tilde{W}_T$ under Q is

$$E_Q\left[e^{\lambda \tilde{W}_T}\right]$$

This can be written as

$$E_Q\left[e^{\lambda \tilde{W}_T}\right] = E_P\left[\frac{dQ}{dP}e^{\lambda \tilde{W}_T}\right]$$

$$E_Q\left[e^{\lambda \tilde{W}_T}\right] = E_P\left[e^{-\mu W_T - \frac{1}{2}\mu^2 T}e^{\lambda(W_T+\mu T)}\right]$$

$$= e^{-\mu\lambda T - \frac{1}{2}\mu^2 T}E_P\left[e^{-(\lambda-\mu)W_T}\right]$$

$$= e^{-\mu\lambda T - \frac{1}{2}\mu^2 T}e^{\frac{1}{2}(\lambda-\mu)^2 T}$$ as W_T ~ N(0, T) under probability measure P

$$= e^{\frac{1}{2}\lambda^2 T}$$

From the uniqueness of the moment generating function of N(0, T), we can conclude that $\tilde{W}_T$ ~ N(0, T) under probability measure Q.

(d) Let W_t be a Standard Brownian Motion under Probability measure P and γ_t be a F - previsible process satisfying the technical condition $E_P\left[\exp(\frac{1}{2}\int_o^T \gamma_s^2 ds)\right] < \infty$. Then, for any fixed time horizon T, there exists a measure Q, equivalent to P, such that the process $\tilde{W}_t = W_t + \int_o^t \gamma_s ds$ is Standard Brownian Motion under Q, with $\frac{dQ}{dP} = e^{-\int_o^T \gamma_s dWs - \frac{1}{2}\int_o^T \gamma_s^2 ds}$.

[10]

4. (a) $$\frac{dA_t}{dt} = aW_t + 2bt$$

$$\frac{dA_t}{dW_t} = 3W_t^2 + at$$

$$\frac{d^2A_t}{dW_t^2} = 6W_t$$

therefore

$$dA_t = (3W_t^2 + at)dW_t + (aW_t + 2bt + 3W_t)dt$$

for A_t to be martingale the drift term needs to be 0 implying

a= -3 and b=0.

$$\frac{dB_t}{dt} = c$$

$$\frac{dB_t}{dW_t} = 2W_t + a$$

$$\frac{d^2B_t}{dW_t^2} = 2$$

therefore

$$dB_t = (2W_t + a)dW_t + (c+1)dt$$

for B_t to be martingale the drift term needs to be 0 implying

c= -1.

So for a = -3, b = 0 and c = -1, both A_t and B_t are martingales.

(b) $A_t = W_t^3 - 3tW_t$

And

$$B_t = W_t^2 - 3W_t - t$$

Then,

$$D_t = (W_t^3 - 3tW_t)(W_t^2 - 3W_t - t)$$

$$\frac{dD_t}{dt} = (W_t^3 - 3tW_t)(-1) + (-3W_t)(W_t^2 - 3W_t - t)$$

$$= (-4W_t^3 + ...)$$

$$\frac{dD_t}{dW_t} = (W_t^3 - 3tW_t)(2W_t - 3) + (3W_t^2 - 3t)(W_t^2 - 3W_t - t)$$

$$= (5W_t^4 + ...)$$

$$\frac{d^2 D_t}{dW_t^2} = (20W_t^3 + ...)$$

Therefore,

$$\begin{aligned} dD_t &= (...)dW_t + ((20W_t^3 + ...) + (-4W_t^3 + ...))dt \\ &= (...)dW_t + (16W_t^3 + ...)dt \end{aligned}$$

There is a drift term in the above expression implying D_t is not a martingale.

(c) Because of the denominator term, the process will always have drift and hence C_t will not be a martingale.

[10]

5. (a) $q = \frac{1.01 - 0.96}{1.04 - 0.96}$

$q = 0.625$

SENSEX

1 October 2010	**1 October 2011**	**1 October 2012**	**1 October 2013**	**1 October 2014**
				11698.59
			11248.64	
		10816		**10798.69**
10000	**10400**		**10383.36**	
		9984		**9968.03**
			9584.64	
				9201.25

Benefit Amount

1 October 2010	**1 October 2011**	**1 October 2012**	**1 October 2013**	**1 October 2014**
				56000
			54000	
		52000		**52000**
	50000		**50000**	
		48000		**48000**
			46000	
				44000

$Value_t = (qf_{u,t} + (1-q) f_{d,t}) / 1.01$

1 October 2010	**1 October 2011**	**1 October 2012**	**1 October 2013**	**1 October 2014**
				56000
			53960.4	
		51955.7		**52000**
	49985.4		**50000.0**	
		48034.5		**48000**
			46039.6	
				44000

The bank should price the bond at Rs49985.40.

(b) Let the bank hold "a" units of the SENSEX and "b" units of the risk free asset. Then,

$10816a + 1.01b = 51955.7$

and

$9984a + 1.01b = 48034.5$

implies

$832a = 3921.2$

Therefore,

$a = 4.713$ and $b = 970.396$

The bank should buy 4.713 unit of SENSEX and invest Rs970.396 in the risk free asset.

(c) $q = \dfrac{1.01 - 0.95}{1.05 - 0.95}$

$q = 0.6$

SENSEX

1 October 2011	**1 October 2012**	**1 October 2013**	**1 October 2014**
			12039.3
		11466	
10400	**10920**		**10892.7**
		10374	
			9855.3

Benefit Amount

1 October 2011	1 October 2012	1 October 2013	1 October 2014
			57500
		55000	
50000	**52500**		**53000**
		50500	
			48500

$Value_t = (qf_{u,t} + (1-q)f_{d,t})/1.01$

1 October 2011	1 October 2012	1 October 2013	1 October 2014
			57500
		55148.51	
	52837.95		**53000**
		50693.07	
			48500

You can sell the bond at Rs.52837.95.

Had you instead invested in the replicating portfolio, the value of the portfolio would be

$10920 \times 4.713 + 1.01 \times 970.396 = 52446.06$

So by selling the bond you make a profit of Rs391.89.

[15]

6. $v_T(t,T) = s(t,T,\Omega_t) = \sigma e^{-a(T-t)}$

Integrating $v_T(t,T)$, we get

$v(t,T) = -\frac{1}{a}\sigma e^{-a(T-t)} + C(t)$ (constant of integration)

Since $v(T,T) = 0$, we have

$C(t) = \frac{\sigma}{a}$

$v(t,T) = \frac{\sigma}{a}[1 - e^{-a(T-t)}] = \sigma B(t,T)$

From Hull-White model, we have

$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$

$$r(t) = -\frac{1}{B(t,T)}[\ln P(t,T) - \ln(A,T)]$$

$$\frac{\partial r}{\partial P} = -\frac{1}{B(t,T)P(t,T)}$$

Using Ito lemma and HJM model, we get the volatility of short rate as:

$$\frac{\partial r}{\partial P} v(t,T,\Omega)P(t,T) = \frac{v(t,T)}{B(t,T)} = \sigma$$

This shows that Hull-White model is consistent with $s(t,T) = \sigma e^{-a(T-t)}$

[7]

7. If the credit spread is s, the current value of the debt is $De^{-(r+s)T}$ (1)

The current value of the debt in Merton's model is $V_0 - E_0$

$$V_0 - E_0 = V_0 - [V_0\Phi(d_1) - De^{-rT}\Phi(d_2)]$$

$$V_0 - E_0 = De^{-rT}\Phi(d_2) + V_0[1 - \Phi(d_1)]$$

$$V_0 - E_0 = De^{-rT}\Phi(d_2) + V_0\Phi(-d_1) \qquad (2)$$

From (1) and (2), we have

$$De^{-(r+s)T} = De^{-rT}\Phi(d_2) + V_0\Phi(-d_1)$$

Substituting $De^{-rT} = LV_0$

$$LV_0 e^{-sT} = LV_0\Phi(d_2) + V_0\Phi(-d_1)$$

$$Le^{-sT} = L\Phi(d_2) + \Phi(-d_1)$$

$$e^{-sT} = \Phi(d_2) + \frac{\Phi(-d_1)}{L}$$

$$-sT = \ln[\Phi(d_2) + \frac{\Phi(-d_1)}{L}]$$

$$s = -\frac{\ln[\Phi(d_2) + \frac{\Phi(-d_1)}{L}]}{T}$$

[10]

8.(a) The delta indicates that when the value of the dollar exchange rate increases by 1 paise (Rs. 0.01), the value of ICICI Bank's position increase by 100,000 X 0.01 = Rs. 1000.

The gamma indicates that when the value of the dollar exchange rate increases by 1 paise (Rs. 0.01), the delta of ICICI Bank's position decreases by 5000 X 0.01 = 50.

(b) For delta neutrality, 100,000 US dollars should be shorted.

(c) When the dollar exchange rate moves up to Rs. 53, we expect that the delta of the bank's position decreases by (53 – 50)5000 = 15,000 so that it becomes 85,000.

(d) To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 15,000 US dollars so that a net $85,000 have been shorted.

(e) When a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset (US dollars) prices. Thus, ICICI Bank is likely to have lost money.

[10]

9.(a) Consider a portfolio (Π) consisting of $\sigma_2 f_2$ of the first derivative and $-\sigma_1 f_1$ of the second derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2 \quad \textbf{(1)}$$

$$d\Pi = (\sigma_2 f_2) df_1 - (\sigma_1 f_1) df_2$$

$$d\Pi = (\sigma_2 f_2)[\mu_1 f_1 dt + \sigma_1 f_1 dz] - (\sigma_1 f_1)[\mu_2 f_2 dt + \sigma_2 f_2 dz]$$

$$d\Pi = (\sigma_2 f_2 \mu_1 f_1 - \sigma_1 f_1 \mu_2 f_2) dt \quad \textbf{(2)}$$

Because this equation does not involve dz, the portfolio is instantaneous riskless. Thus, it must earn risk-free rate.

$$d\Pi = r\Pi dt \quad \textbf{(3)}$$

From (1), (2) and (3), we have

$$(\sigma_2 f_2 \mu_1 f_1 - \sigma_1 f_1 \mu_2 f_2) dt = r[\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2] dt$$

$$\sigma_2 f_2 \mu_1 f_1 - \sigma_1 f_1 \mu_2 f_2 = r\sigma_2 f_2 f_1 - r\sigma_1 f_1 f_2$$

$$\sigma_2 \mu_1 - \sigma_1 \mu_2 = r\sigma_2 - r\sigma_1$$

Dividing both side by $\sigma_1 \sigma_2$, we get

$$\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2} = \frac{r}{\sigma_1} - \frac{r}{\sigma_2}$$

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

(b) $$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \sigma_2$$

$$\mu_1 = \sigma_1 \sigma_2 + r; \quad \mu_2 = \sigma_2^2 + r$$

$$df_1 = [r + \sigma_1 \sigma_2] f_1 dt + \sigma_1 f_1 dz$$

$$df_2 = [r + \sigma_2^2] f_2 dt + \sigma_2 f_2 dz$$

Using Ito-Lemma, we have

$$d \ln f_1 = [r + \sigma_1 \sigma_2 - \frac{\sigma_1^2}{2}] dt + \sigma_1 dz$$

$$d \ln f_2 = [r + \sigma_2^2 - \frac{\sigma_2^2}{2}] dt + \sigma_2 dz$$

$$d(\ln f_1 - \ln f_2) = (\sigma_1 \sigma_2 - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2}) dt + (\sigma_1 - \sigma_2) dz$$

$$d \ln(\frac{f_1}{f_2}) = -\frac{(\sigma_1 - \sigma_2)^2}{2} dt + (\sigma_1 - \sigma_2) dz$$

Assume $G = \frac{f_1}{f_2}$ and $x = \ln(\frac{f_1}{f_2}) = \ln G$.

$$G = e^x$$

$\frac{\partial G}{\partial x} = e^x = G$, $\frac{\partial G}{\partial t} = 0$, and $\frac{\partial^2 G}{\partial x^2} = e^x = G$

From Ito lemma, we have

$$dG = [-G \frac{(\sigma_1 - \sigma_2)2}{2} + \frac{1}{2} G(\sigma_1 - \sigma_2)^2] dt + (\sigma_1 - \sigma_2) G dz$$

$$dG = (\sigma_1 - \sigma_2) G dz$$

$$d\left(\frac{f_1}{f_2}\right) = (\sigma_1 - \sigma_2)\left(\frac{f_1}{f_2}\right)dz$$

This show that $\frac{f_1}{f_2}$ is a martingale.

[13]

10. (a) MARKS WILL ALSO BE GIVEN FOR CONTINUOUSLY COMPOUNDED RATES.

Let P(i) be the price of zero coupon bong with duration i. Then the spot rate at time t is given by

$$r(t) = P(t)^{-1/t} - 1$$

For t=1, the forward rate is equal to spot rate,

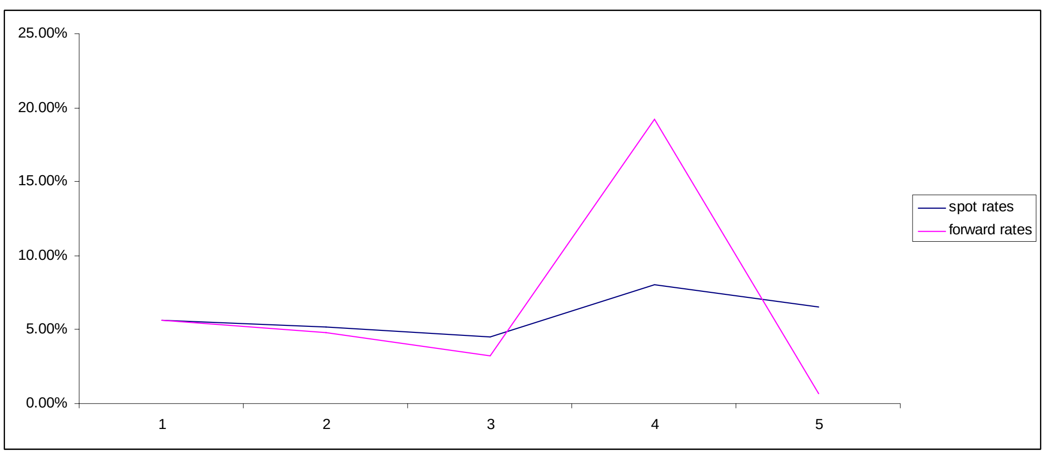
$$r(t) = f(t)$$

For t>1, the forward rate is given by

$$f(t) = \frac{P(t-1)}{P(t)} - 1$$

Using these relationships,

t	P(t)	r(t)		f(t)	
		Formula	Value	Formula	Value
1	94.7%	$(94.7\%)^{-1}-1$	5.6%	100.0%/94.7% - 1	5.6%
2	90.4%	$(90.4\%)^{-1/2}-1$	5.2%	94.7%/90.4% - 1	4.8%
3	87.6%	$(87.6\%)^{-1/3}-1$	4.5%	90.4%/87.6% - 1	3.2%
4	73.5%	$(73.5\%)^{-1/4}-1$	8.0%	87.6%/73.5% - 1	19.2%
5	73.0%	$(73.0\%)^{-1/5}-1$	6.5%	73.5%/73.0% - 1	0.7%



(b) (i)

t	Payment as a % of the face value	Present value factor	Present value of payments as a % of the face value
1	8%	94.7%	7.576%
2	8%	90.4%	7.232%
3	8%	87.6%	7.008%
4	8%	73.5%	5.880%
5	113%	73.0%	82.490%
Total			110.186%

The price of the 5 year government bond redeemed at a premium of 5% and paying a coupon of 8% is 110.2% of the face value.

(ii)

t	Payment as a % of the face value	Present value factor @ of 6% per annum	Present value of payments as a % of the face value
1	8%	94.3%	7.544%
2	8%	89.0%	7.120%
3	8%	84.0%	6.720%
4	8%	79.2%	6.336%
5	113%	74.7%	84.411%
Total			112.131%

At 6% the price of bond is 112.1% of the face value implying that the IRR of the 5 year government bond is higher than 6%. Thus it is better to buy the bond rather than investing in the term deposit.

(iii) Present value of annuity amounts is given by

$$PVAA(t) = x \times SP \times \sum_{1}^{5} P(t)$$

We want

$$PVAA(t) = 0.8 \times SP$$

Hence,

$$0.8 = x \times \sum_{1}^{5} P(t)$$

$$0.8 = x \times 4.192$$

$$x = 19.08\%$$

The bank should pay 19.08% of the single premium, so that it can make a profit margin of 20%.

(iv) Let the equal profits each year be a% of the single premium.

Present value of profits is given by

$$PVP(t) = a \times SP \times \sum_{1}^{5} P(t)$$

We want

$$PVP(t) = 0.2 \times SP$$

Hence,

$$0.2 = a \times \sum_{1}^{5} P(t)$$

$0.2 = a \times 4.192$

$a = 4.77\%$

The total outgo plus profit emerging for the bank is 23.85% of the single premium. The following should be the investment in each zero coupon bond.

t	Amount to be realised as a % of single premium	Present value factor	Initial Investment as a percentage of single premium
1	23.85%	94.7%	22.59%
2	23.85%	90.4%	21.56%
3	23.85%	87.6%	20.89%
4	23.85%	73.5%	17.53%
5	23.85%	73.0%	17.41%
Total			100.00%

[15]

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