CT8- SOLUTION

1 The investor's utility is depicted through a power utility function.

(i) For a risk averse investor, the utility function condition is

$$U''(w) < 0$$

For power utility function we have

$$U'(w) = w^{\gamma-1}\text{, and } U''(w) = (\gamma - 1)w^{\gamma-2}$$

Thus, to model the behaviour of a risk averse investor we should have

$\{\gamma < 1; \gamma \neq 0\}$.

(ii) Iso-elastic utility functions display constant elasticity of marginal utility (with respect to wealth) as wealth increases. Such utility functions exhibit constant relative risk aversion, ie:

$$R(w) = \text{constant and } R'(w) = 0$$

where R(w) is defined as

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

For power utility function

$$R(w) = (1 - \gamma) \text{ and } R'(w) = 0$$

Hence the power utility function is iso-elastic.

(iii) α is the amount invested in Z.

The following outcomes are possible

$$Poor: w = 0.9\alpha + 1 - \alpha = 1 - 0.1\alpha$$

$Good: w = 1.2\alpha + 1 - \alpha = 1 + 0.2\alpha$

The expected utility of outcome defined by O(α) is given by

$O(\alpha) = 0.5U(1 - 0.1\alpha) + 0.5U(1 + 0.2\alpha)$

$$= 0.5\frac{(1-0.1\alpha)^{\gamma}-1}{\gamma} + 0.5\frac{(1+0.2\alpha)^{\gamma}-1}{\gamma}$$

We need to find the value of α for which O(α) attains its maximum value.

To do this we take the derivative of *O*,

$$O'(\alpha) = \frac{0.5}{\gamma}(-0.1\gamma(1-0.1\alpha)^{\gamma-1} + 0.2\gamma(1+0.2\alpha)^{\gamma-1})$$

set it equal to 0

$$\frac{0.05}{\gamma}\gamma((1-0.1\alpha)^{\gamma-1} - 2(1+0.2\alpha)^{\gamma-1}) = 0$$

and solve for α

$$2 = \left(\frac{1+0.2\alpha}{1-0.1\alpha}\right)^{1-\gamma}$$

$$\alpha = \left(\frac{X-1}{0.2+0.1X}\right) \; where \; X = 2^{\frac{1}{1-\gamma}}$$

Hence, the amount of wealth invested in Z if $\gamma = -3$ is 0.5933.

(iv) The amount of wealth invested in Z for various values of γ can be calculated by substituting these values in the expression for α above

Values of γ	Amount invested in Z
-1	1.2132
-3	0.5933
-5	0.3922
-7	0.2929

As γ **decreases,** the amount invested in Z (risky security) reduces i.e. the investor becomes more risk averse.

(v) The utility function is iso-elastic.

This means that an x% increase in wealth induces an x% increase in the holding of the risky asset.

Therefore, if the amount of wealth invested in risky asset is 0.5933 when initial wealth is 1 unit, the amount of wealth invested in risky asset would be 296.6366 if the initial wealth is 500 units.

[Total 20]

2 (i) Difficulties faced when testing the EMH *(any four)*

- It is difficult to define publicly available information.
- It is difficult to determine when, precisely, information arrives.
- EMH does not specify how information is priced, so, very difficult to test.
- There is no universally agreed definition of risk, and no perfectly accurate way of measuring it. Hence difficult to conclude if out-performance of an investment strategy contradicts EMH or not.
- Even if the market is efficient, pure chance is going to throw up some apparent examples of mispricings.
- Testing requires making implicit/explicit assumptions which are open to criticism. For example choice of discount rate, normality or returns, stationary of time series, estimates of variance and covariance etc.

(ii) *Testing the strong form EMH*

This is problematic, as it requires the researcher to have access to information that is not in the public domain.

However, studies of directors' share dealings suggest that, even with inside information, it is difficult to out-perform.

Testing the weak form EMH

Using price history to try and forecast future prices, often using charts of historical data, is called technical or chartist analysis. Studies have failed to identify a difference between the returns on stocks using technical analysis and those from purely random stock selection after allowing for transaction costs.

No credible challenge has emerged to the EMH in its weak form.

Testing the semi-strong form EMH

The semi-strong form of the EMH has been where research has concentrated and in particular focused on tests of informational efficiency and volatility tests.

Informational efficiency

Many studies show that the market over-reacts to certain events and under-reacts to other events. The over/under-reaction is corrected over a long time period. If this is true then traders could take advantage of the slow correction of the market, and efficiency would not hold.

Volatility tests

Shiller first formulated the claim of "excessive volatility" into a testable proposition in 1981. He found strong evidence that the observed level of volatility contradicted the EMH. However, subsequent studies using different formulations of the problem found that the violation of the EMH only had borderline statistical significance.

[Total 8]

3 (i) Let R_X, R_Y, R_Z denote random return on securities X, Y and Z respectively.

We know the following relationship holds between correlation, covariance and standard deviation of random variables

$$\rho_{i,j} = \frac{Cov(i,j)}{\sigma_i \sigma_j} \Rightarrow Cov(i,j) = \rho_{i,j}\sigma_i\sigma_j$$

$Cov(R_X, R_Y) = 0.75*0.45*0.3 = 0.10125$
$Cov(R_Y, R_Z) = 0.2*0.3*0.15 = 0.009$
$Cov(R_X, R_Z) = -0.1*0.45*0.15 = -0.00675$

Hence the covariance matrix is give by

$$\begin{bmatrix} 0.2025 & 0.10125 & -0.00675 \\ 0.10125 & 0.09 & 0.009 \\ -0.00675 & 0.009 & 0.0225 \end{bmatrix}$$

(ii) The security market line is give by

$$E_i = r + \beta_i(E_M - r)$$

where:

E_i is the expected return on security i
r is the risk free rate of interest
β_i is the beta factor of security i defined as $Cov(R_i, R_M)/V_M$
E_M is the expected return on market portfolio

The market portfolio under CAPM consists of risky securities in proportion to their market capitalisation.

$$E_M = \frac{1}{6}(3E_X + 2E_Y + E_Z) = 30\%$$

$Cov(R_i, R_M) = \frac{1}{6}\left(3\ Cov(R_i, R_X) + 2\ Cov(R_i, R_Y) + Cov(R_i, R_Y)\right)$, i=X,Y,Z,M
So,

$$Cov(R_X, R_M) = 0.1339; Cov(R_Y, R_M) = 0.0821; Cov(R_Z, R_M) = 0.0034$$

and

$$V_M = \frac{1}{6}\left(3\,\mathrm{Cov}(R_M, R_X) + 2\,\mathrm{Cov}(R_M, R_Y) + \mathrm{Cov}(R_M, R_Z)\right) = 0.094875$$

Now the values of beta factor can be derived.

$$\beta_X = \frac{0.1339}{0.094875} = 1.4111;\ \beta_Y = 0.8656;\ \beta_Z = 0.0356$$

Substituting these values in the equation for respective security market lines and solving for expected return on securities yields the following results:

$$E_X = 38.22\%;\ E_Y = 27.31\%;\ E_Z = 10.71\%$$

(iii) Following are some of the problems involved in estimating parameters for asset pricing models

- **data availability**: there may not always be as much data available as required to achieve a certain minimum level of statistical error in estimates.
- **data errors**: even when sufficient volumes of data is available, it may not be accurately captured limiting its use.
- **outliers**: there may be a large number of outliers in the data which do not fit the model; it is difficult to determine the course of action under such circumstances.
- **stationarity of underlying time series**: a modeler may not be justified in accepting a model simply because the model is found to be good fit based on a suitable array of tests. Many of these tests (for example, tests of stationarity) have notoriously low power, and therefore may not reject incorrect models.
- **the role of economic judgment**: The objective of economic models is to simplify reality by imposing certain stylized facts about how markets would behave in an ideal world. The modeling process relies heavily on the skill and judgment of the modeler.

[Total 15]

4 (i) Ito's lemma states that

$$dV(t, X_t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial X_t}dX_t + \frac{1}{2}\frac{\partial^2 V}{\partial X_t^2}(dX_t)^2$$

We have

$$S_t = S_0 e^{\mu t + \sigma B_t} = f(t, B_t)$$

using Ito's lemma

$$dS_t = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial^2 S}{\partial B_t^2}(dB_t)^2$$

$$\frac{\partial S}{\partial t} = \mu S_0 e^{\mu t + \sigma B_t}; \; \frac{\partial S}{\partial B_t} = \sigma S_0 e^{\mu t + \sigma B_t}; \; \frac{\partial^2 S}{\partial B_t^2} = \sigma^2 S_0 e^{\mu t + \sigma B_t}$$

and we know that

$$(dB_t)^2 = dt$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion and μ and σ are constants.

Substituting these values we get

$$dS_t = \mu S_t dt + \sigma S_t dB_t + \frac{1}{2}\sigma^2 S_t dt = S_t \left\{ \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dB_t \right\}$$

(ii) For a standard Brownian motion we know that

$$B_t \sim N(0, t) \Rightarrow \sigma B_t \sim N(0, \sigma^2 t) \Rightarrow e^{\sigma B_t} \sim \mathrm{LogN}(0, \sigma^2 t)$$

Hence,

$$E[e^{\sigma B_t}] = E[\mathrm{LogN}\big((0, \sigma^2 t)\big] = e^{\frac{\sigma^2 t}{2}}$$

$$\mathrm{Var}[e^{\sigma B_t}] = \mathrm{Var}[\mathrm{LogN}\big((0, \sigma^2 t)\big] = e^{\sigma^2 t}(e^{\sigma^2 t} - 1)$$

$$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}] = S_0 e^{\mu t} E[e^{\sigma B_t}] = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$$

$$\mathrm{Var}[S_t] = Var[S_0 e^{\mu t + \sigma B_t}] = S_0^2 e^{2\mu t} \mathrm{Var}[e^{\sigma B_t}] = S_0^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1)$$

[Total 7]

Sol 5

(i) A theta of – 0.1 means that with Δt years pass and no change in either the stock price or its volatility or risk-free interest rate, the value of the option will decline by $0.1\Delta t$

(ii) Black Scholes PDE states;

$$r\Pi = \Theta + \Delta(r-q)S_t + \frac{\sigma^2}{2} * \Gamma * S_t^2$$

for a delta neutral portfolio

$$r\Pi = \Theta + \frac{\sigma^2}{2} * \Gamma * S_t^2$$

In the above R.H.S of above equation both increase and decrease in the asset price will have same impact over a short period of time. Hence, if the gamma is –ve and large, portfolio value will decrease.

Being an option writer will imply person will lose significant amount of money if there is a large movement.

(iii) S = 555
K = 545
q= 3%;
r = 8%;
σ = 25% and
T = 4/12

$$\text{Rho} = -K*T*\exp^{(-r*T)}\ \Phi(-d_2)$$

Where $d_2 = [\log(S/K)+(r-q-\sigma^2/2)*T] / \sigma\sqrt{T} = 0.169272$

$$\Phi(-0.167292) = 0.432791$$

$$\text{Rho} = -545*0.333*0.97368*0.43357 = -76.5546$$

(iv) Rho of a portfolio of options is the rate of change of the value if the portfolio with respect to the interest rate. Here for 2% change in the risk free rate, value of the option decreases by 1.53385
(0.02*76.5546 = 1.5311)

[Total 12]

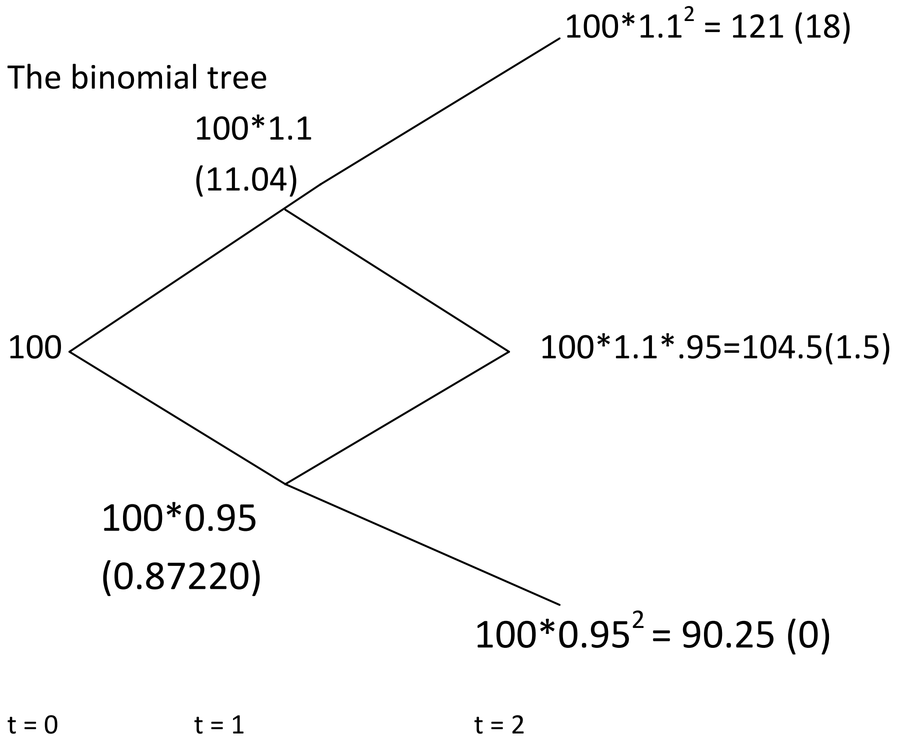
Sol 6 (i)

In a *recombining* binomial tree *u* and *d* , the proportionate increase and decrease in the underlying security price at each step, are assumed constant throughout the tree. As a result the security price after a specified number of up- and down- movements is the
same, irrespective of the order in which the movements occurred.

In a *non-recombining* binomial tree the values of *u* and *d* can change at each stage. As a result, each node in the tree will in general generate two new nodes, making the tree much larger than a combining tree. Consequently, an *n* -period tree will have 2^n , rather than just *n* +1, possible states at time *n* .

Sol6 (ii)

The binomial tree



Value of the option in brackets on the tree

S_i = Stock price at node i

C_i = Option Price at node i

U = 1.1 ; d = 0.95 ; $S_1 = uS_0$

At t = 2 we know the option value with certainty

So at t = 1 upper node we need to calculate the holdings of cash and stock which will replicate the value of option at t = 2

Let Cash holding is ψ_k

Stock Holding ϕ_k

$$C_{up} = \phi_{now} S_{now} u + \psi_{now} e^r$$

$$C_{down} = \phi_{now} S_{now} d + \psi_{now} e^r$$

Hence

$18 = \phi_1 * 121 + \psi_1 e^{0.04}$ If stock goes up

$1.5 = \phi_1 * 104.5 + \psi_1 e^{0.04}$ If stock goes down

$$\phi_{now} = \frac{C_{up} - C_{down}}{S_{now}(u - d)}$$

$$\psi_{now} = e^{-0.04} \frac{\left\{ C_{down} u - C_{up} d \right\}}{(u - d)}$$

$\phi_1 = 1$; $\psi_1 = -98.96$

Hence the value = 1 * 110 – 98.96 = 11.04

Similarly

$\phi_2 = 0.10526$; $\psi_2 = -9.12750$

The value = 95*0.10526-9.12750 = 0.87220

Hence the Replicating Portfolio therefore at t = 0 is

$$\phi_0 = \frac{11.04 - 0.8722}{110 - 95}$$

$= 0.67785$

$$\psi_0 = e^{-0.04} \left\{ \frac{0.8722*1.1 - 11.04*0.95}{1.1-0.95} \right\}$$

$= -61.03306$

Hence value of option at t = 0 is

0.67785*100 – 61.03306 = 6.75194

[Total 8]

Sol7

(i) Six Factors affecting option price are:-

Current Stock Price; Strike Price; Risk-free Interest Rate; Volatility of the stock price; Time to Expiry; Dividends expected during the life of the option

(ii)

	European Call	American Put
Current Stock Price	+	-
Strike Price	-	+
Risk-free Interest Rate	+	-
Volatility of the stock price	+	+
Time to Expiry	?	+
Dividends	-	+

+ indicates that an increase in variable causes the option price to increase

- indicates that an increase in variable causes the option price to decrease

? Indicates relationship is uncertain

(iii)

Range of stock Price	Payoff from call	Payoff from Put	Total
S_t<=K	0	K-S_t	K-S_t
S_t>K	S_t-K	0	S_t-k

For this situation

Bought a Call for Rs6 and Put for Rs 4. Hence total cost is

Range of	Payoff from	Payoff from	Total	Profit

stock Price	call	Put		
S_t<=60	0	60-S_t	60-S_t	50- S_t
S_t>60	S_t-60	0	S_t-60	S_t -70

The above strategy shows that incase of large movement either ways it will result in profit. Only if Stock Price moves within 50 and 70 then there would be a loss.

[Total 12]

Sol 8

(i)

The three main limitations of one-factor models are:

1. They are consistent with perfect correlation of bond prices, which is inconsistent with the empirical evidence
2. They assume a constant volatility of interest rates, which is again inconsistent with the empirical evidence
3. They may be unsuitable for pricing complex options, *eg* those whose payoffs depend on more than one interest rate.

(ii)

The equations defining the two models are similar:

- Both are continuous-time Markov models of the short rate of interest
- Ito processes defined by a stochastic differential equation
- one-factor models (ie they incorporate only one source of randomness)
- usually defined in terms of a standard Brownian motion under the risk-neutral probability measure.

- Both models imply that the short rate is mean-reverting.
- Both models imply that the future short rate has a normal distribution.
- Both models allow negative values for the short rate.
- Both models are mathematically tractable, although the Hull & White model is slightly more complicated algebraically.

The key difference is that the Vasicek model is time-homogeneous (with constant m), whereas the Hull & White model is not (with time-dependent m).

The Hull & White model has to be calibrated (ie a function has to be selected for m (t)) to match the current pattern of bond prices.

(iii)

Vasicek Model Standard deviation will be 1%. CIR model S.D of short rate is proportional to the square root of the short rate. When the short rate increases from 4% to 8% the S.D of the short rate increases from 1% to 1.414%

i.e $\sqrt{.08} / \sqrt{.04} = 1.414$

[Total 10]

9) Recovery rate for a bond is normally defined as the bond's market value immediately after a default as a percent of its face value

(ii) probability = s/ (1-R)

Where s – spread of the corporate bond yield and R is expected recovery rate.

= 0.02 / (1-0.4) = 3.33%

(iii) Assume no other debt, frictionless markets, perfect information, black scholes type model for the movement of the assets

Use black scholes to calculate the value of a call option based on the value of the assets exceeding a strike of 10 after 10 years.
S = 35 crores = Current market value of the company's assets.
K = 1 crore = Maturity value of the debt

r = 5%;

σ = 80% and

T = 10

Value of the

Equal to equity of firm = *E* = Rs. 34.54415 crores

Value of bond is *B* = 35 – *34.54415* = 0.4558494

Now The difference between the bond yield of 5.92% and the default-free rate of 5% is called the *credit spread*.
Solve
$10*\exp^{(-10\, r_b)} = B$

$$r_b = \frac{-1}{10} \ln\ 0.4558494 = 7.856\%$$

r_b – 5% = 2.856% p.a continuously compounded is the credit spread

[Total 8]

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