Institute of Actuaries of India

Subject CT4 – Models

November 2011 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) The process adheres to the Markov Property since the probability of moving on to the next state does not depend on the history of the process prior to entering that state.

T_k has a discrete state space (0,1,2…8) and discrete time set since the value of the process is recorded every week. The process, therefore, is a Markov chain.

ii)

a) The probability that A R Rehman will compose all 8 tracks in 8 weeks is:

$0.80^8 = 0.1678$

b) A R Rehman will exactly take 12 weeks to compose the music where:

- He has finished exactly 7 tracks in first 11 weeks; and
- He finished the last track in the 12^{th} week.

The probability for finishing exactly 7 tracks in first 11 weeks is:

$$\binom{11}{7} 0.20^4 x\, 0.80^7 = 0.1107$$

The probability of finishing the last track in 12^{th} week is 80%.

Therefore, the probability that A R Rehman will take exactly 12 weeks to compose all 8 tracks is 0.1107 x 80%=0.0886.

iii) Define m_k to be the expected time in weeks for all the tracks to be composed, given that k tracks have been composed.

We have:

$$m_k = 1 + 0.2m_k + 0.8m_{k+1}$$

$$\therefore m_k = \frac{1}{0.80} + m_{k+1}$$

By definition, $m_8 = 0$.

$$\therefore m_7 = \frac{1}{0.80}$$

$$\therefore m_6 = \frac{1}{0.80} + \frac{1}{0.80} = \frac{2}{0.80}$$

$$\therefore m_5 = \frac{1}{0.80} + \frac{1}{0.80} + \frac{1}{0.80} = \frac{3}{0.80}$$

In general:

$$m_k = \frac{8-k}{0.80}$$

$$\therefore m_0 = \frac{8}{0.80} = 10 \text{ weeks}$$

The expected number of weeks that A R Rehman will take to compose all 8 tracks is 10.

iv) Using the expression in (iii):

a) The expected number of weeks for remaining 5 tracks is given by:

$$m_3 = \frac{8-3}{0.80} = 6.25 \text{ weeks}$$

b) Since the process follows the Markov Property, it does not matter whether A R Rehman has taken 3 weeks to compose 3 tracks or 10 weeks to compose 3 tracks. The expected number of weeks for remaining 5 tracks would be same as (a) above – 6.25 weeks.

[9]

Solution 2:

i) Commonly used models of mortality are based on the assumption that each individual live has identical mortality characteristics, which is rarely true in practice. As a result, the estimates of mortality rates are averaged across the entire population under investigation. This could be a particular problem for an insurance company who wishes to set premiums that accurately reflect the riskiness of each individual policyholder.

Sub-dividing data according to characteristics - known from experience to have a significant impact on mortality - such as age, sex, smoker status etc have the advantage of providing a relatively more accurate estimate of mortality for homogenous groups and thus improving the confidence of estimating the individual's riskiness.

One disadvantage of sub-dividing data is that some degree of heterogeneity will still remain and sub-division using many factors may results in much smaller populations in each class, making the statistics more difficult.

ii) Calculating contribution to exposed to risk

a) Central exposed to risk

Assume that the day of entry is counted in the exposed to risk but the day of exit is not.

Assume that the age label used in the investigation is "age last birthday".

Assume that policy ceases on the date of death of John and Angelina is censored from the investigation on the same date.

Based on these, we can consider the central exposed to risk by year for each person:

	Date of Birth	start	end	age	day count
John	18-Dec-63	27-May-06	18-Dec-06	42	205
		18-Dec-06	18-Dec-07	43	365
		18-Dec-07	18-Dec-08	44	366
		18-Dec-08	18-Dec-09	45	365
		18-Dec-09	01-Jul-10	46	195
Angelina	04-Jun-75	27-May-06	04-Jun-06	30	8
		04-Jun-06	04-Jun-07	31	365
		04-Jun-07	04-Jun-08	32	366
		04-Jun-08	04-Jun-09	33	365
		04-Jun-09	04-Jun-10	34	365
		04-Jun-10	01-Jul-10	35	27

b) Initial exposed to risk

John receives the same exposure in the initial exposed to risk at all ages except at age 46 in which he receives 365 days (even though this extends beyond the period of investigation).

Angelina receives the same exposure in the initial exposed to risk at all ages. She still receives 27 days in age 35 since she did not die – initial exposed to risk only gets counted to the year-end if the person leaves by the decrement rate being investigated.

[9]

Solution 3:

i) Since the particles behave independently of one another, the likelihood function can be written as:

$$L = \prod_{i=1}^{N} P(D_i = d_i)$$

$$\therefore L = \prod_{Decayed} P(D_i = 1) \;\; X \prod_{Did\ not\ decay} P(D_i = 0)$$

Now, consider an i^{th} neutrino that did not decay. This particle was observed between the time a_i and t_i; and the force of decay is μ. Therefore:

$$P(D_i = 0) = e^{-\mu(t_i - a_i)}$$

Now, consider an i^{th} neutrino that decayed away. This particle was observed between the time a_i and t_i; and then decayed. Therefore:

$$P(D_i = 1) = \mu e^{-\mu(t_i - a_i)}$$

Assuming that out of N neutrinos, d decayed away:

$$L = \prod_{Decayed} \mu e^{-\mu(t_i - a_i)} \; x \prod_{Did\ not\ decay} e^{-\mu(t_i - a_i)}$$

$$\therefore L = \mu^d \prod_{i=1}^{N} e^{-\mu(t_i - a_i)}$$

$$\therefore L = \mu^d x \exp\left\{\sum_{i=1}^{N} -\mu(t_i - a_i)\right\}$$

$\therefore L = \mu^{d} e^{-\mu v}$

Where the waiting time over all neutrinos is indicated by:

$$v = \left\{ \sum_{i=1}^{N} (t_i - a_i) \right\}$$

The likelihood function can be maximized to solve for μ. Alternatively, the log-likelihood function can be maximized to solve for μ.

The log-likelihood function is:

$$\ln L = d \ln \mu - \mu v$$

Differentiating w.r.t. μ:

$$\frac{d \ln L}{d\mu} = \frac{d}{\mu} - v$$

Setting this to zero:

$$\mu = \frac{d}{v}$$

Checking the second derivative:

$$\frac{dd^2 \ln L}{d\mu^2} = \frac{-d}{\mu^2} < 0$$

Therefore, the maximum likelihood estimate is given by:

$$\hat{\mu} = \frac{d}{v}$$

ii) The asymptotic distribution of the maximum likelihood estimator $\hat{\mu}$ is $N\left(\mu, \frac{\mu}{E(V)}\right)$; where $E(V)$ denotes the expected waiting time for all the particles under observation.

iii) An approximate 95% confidence interval for $\hat{\mu}$ is:

$$\hat{\mu} \pm 1.96 \sqrt{var(\hat{\mu})}$$

We have:

$$\hat{\mu} = \frac{16}{1{,}000}$$

Also:

$$var(\hat{\mu}) = \frac{\hat{\mu}}{v}$$

$$var(\hat{\mu}) = \frac{16}{1{,}000{,}000}$$

Therefore, the approximate 95% confidence interval for μ is:

$$\frac{16}{1{,}000} \pm 1.96\sqrt{\frac{16}{1{,}000{,}000}}$$

i.e. $\{0.00816, 0.02384\}$

[10]

Solution 4:

i) We can re-write the journal data for manufacturing defects of mobile phones, using terminology for mortality investigations as follows:

t	Defects	Censored	Lives
			100
1		7	93
2	1	2	90
3			90
4		2	88
5	2	3	83
6			83
7			83
8	1	1	81
9			81
10			81
11	2		79
12			79

(Note that even though no calls were made on 30/06, it was confirmed on 31/07 that there were no manufacturing defects for t=6 and t=7)

Assuming that phone sets which were censored at any time t were at risk of defects at time t, then we can calculate the required statistics as follows, where t is the time since purchase of a new phone, calculated in months:

j	t_j	n_j	d_j	$\hat{\lambda}_j$	$\hat{\Lambda}(t)$	$\hat{S}(t)$
0	0	100			0.00000	1.0000
1	2	93	1	0.01075	0.01075	0.9893
2	5	88	2	0.02273	0.03348	0.9671
3	8	83	1	0.01205	0.04553	0.9555
4	11	81	2	0.02469	0.07022	0.9322

Where,

t_j is the j[th] time of defect

n_j is the number of phones available for defect at each time of defect

d_j is the number of defects at each time of defect

$\hat{\lambda}_j = \dfrac{d_j}{n_j}$ is the estimates of the discrete hazard rates at each time of defect

$\hat{\Lambda}(t)$ is the estimate of cumulative hazard function

$\hat{S}(t) = \exp(-\hat{\Lambda}(t))$ is the estimate of the survival function

From the above, we can see that $\hat{S}(t > 11) = 0.9322$

Therefore, the probability of getting defective in one year is estimated to be 1-0.9322 = 6.78%.

ii) From part (i), the probability of getting defective in one year = 6.78%

Therefore, the expected cost of providing the warranty is equal to:

6.78% * 10,000 phones * Rs25,000 = Rs1,69,52,707 ~ Rs1.7 crores

An approximate 95% confidence interval for the integrand hazard function is:

$\hat{\Lambda}_t \pm 1.96\sqrt{\text{var}(\tilde{\Lambda}_t)}$

Where,

$$Var\left[\tilde{\Lambda}_t\right] \approx \sum_{t_j \le t} \frac{d_j(n_j - d_j)}{n_j^3}$$

We can calculate this as follows:

j	t_j	n_j	d_j	$\text{var}(\tilde{\Lambda}_t)$	$\sum_{t_j \le t}$
0	0	100		0.000000	0.000000
1	2	93	1	0.000114	0.000114
2	5	88	2	0.000252	0.000367
3	8	83	1	0.000143	0.000510
4	11	81	2	0.000297	0.000807

Therefore, an approximate 95% confidence interval for integrand hazard function for t>12 is:

$$0.07022 \pm 1.96 \times \sqrt{0.000807}$$

= (0.1259, 0.0145)

And an approximate 95% confidence interval for the survival function for t>12 is equal to (0.8817, 0.9856)

…implying an approximate 95% confidence interval for probability of getting defective in 12 months equal to (11.83%, 1.44%)

…and an approximate 95% confidence interval for the estimated cost of providing replacement warranty equal to (Rs2,95,77,660 ; Rs36,04,645)

iii) The finance director's crude estimate does not allow for data that has been lost to the investigation. For instance, we do not have any information about the phones purchased by the seven individuals who provided incorrect contact details. Similarly, we do not have subsequent information in respect of those who requested not to be disturbed in the future or changed their contact details. It is possible that phones purchased by these individuals may have also developed manufacturing defects within the first year of purchase. Therefore, the crude rates not allowing for censoring introduces a downward bias in the finance director's estimates. Nelson-Aalen estimate allows for censored data by calculating estimates of the discrete hazard rates at each time of death as number of deaths divided by is the number of lives available to die at each time of death. The denominator in this calculation is reduced by the number of lives censored by the time of death. Therefore, the estimate of probability of defects in the first year will be higher using this approach (since it allows for the possibility of censored lives dying as well).

[11]

Solution 5:

i)

a) It is possible to have a Markov chain that has more than one stationary distribution. ***TRUE***

b) It is possible to have a Markov chain that has no stationary distribution. ***TRUE***

c) A Markov chain with a finite state space has at least one stationary probability distribution. ***TRUE***

d) An irreducible Markov chain with a finite state space has a unique stationary probability distribution. ***TRUE***

ii) Let *S* be the state space. We say that $\{\pi_j | j \in S\}$ is a stationary probability distribution for a Markov chain with transition matrix P if the following hold for all $j \in S$:

- $\pi_j = \sum_{i \in S} \pi_i P_{ij}$

- $\pi_j \geq 0$

- $\sum_{j \in S} \pi_j = 1$

(Award only half a mark where a wordy description is provided instead of appropriate mathematical expressions.)

iii) Since this is an irreducible Markov chain with finite state space, there is only one stationary distribution exists for this process.

Define $[\pi_1, \pi_2, \pi_3]$ to be the stationary distribution.

We have:

$$[\pi_1, \pi_2, \pi_3] = [\pi_1, \pi_2, \pi_3]\,P$$

$$\therefore [\pi_1, \pi_2, \pi_3] = [\pi_1, \pi_2, \pi_3]\begin{bmatrix} 1/8 & 1/2 & 3/8 \\ 0 & 1/4 & 3/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

We have:

$$\pi_1 = \frac{1}{8}\pi_1 + \frac{1}{4}\pi_3 \qquad \cdots\cdots\cdots (1)$$

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\pi_3 \qquad \cdots\cdots\cdots (2)$$

$$\pi_3 = \frac{3}{8}\pi_1 + \frac{3}{4}\pi_2 + \frac{1}{4}\pi_3 \qquad \cdots\cdots\cdots (3)$$

From equation (1):

$$\pi_3 = \frac{7}{2}\pi_1$$

Substituting the value for π_3 in equation (2):

$$\pi_2 = \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{2}\cdot\frac{7}{2}\pi_1$$

$$\therefore 4\pi_2 = 2\pi_1 + \pi_2 + 7\pi_1$$

$$\therefore 3\pi_2 = 9\pi_1$$

$$\therefore \pi_2 = 3\pi_1$$

We know that:

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\therefore \pi_1 + 3\pi_1 + \frac{7}{2}\pi_1 = 1$$

$$\therefore \pi_1 = \frac{4}{30}$$

$$\pi_2 = 3\pi_1 = \frac{12}{30}$$

And

$$\pi_3 = \frac{7}{2}\pi_1 = \frac{14}{30}$$

iv) The easiest way to validate the solution is to multiply the stationary distribution with the transition matrix – the resulting distribution should be the stationary distribution itself.

In addition, the solution can also be validated by checking whether:

a) the probabilities within the stationary distribution sums to 1; or
b) all the working equations tally when the values of π_i are replaced by solved values.

[11]

Solution 6:

i)

The mortality rates, as evident from the large experiences, are believed to vary smoothly with age; therefore the crude estimate of mortality at any age carries information about the mortality rates at adjacent ages.

By smoothing the experience, we can make use of data at adjacent ages to improve the estimate at each age.

This reduces the sampling (or random) errors.

This mortality experience may be used in financial calculations. Irregularities, jumps and anomalies in financial quantities (such as premium rates under life insurance contracts) are hard to justify to customers.

The main limitation in mortality investigations that graduation will not be able to overcome is to remove any bias in the data arising from faulty data collection or otherwise.

ii)

a) Cumulative deviations test

The null hypothesis is:

H_0 = the standard table rates are the true underlying mortality rates for the term assurance policyholders

We first calculate the individual deviations using the formula:

$$z_x \approx \frac{\theta_x - E_x q_x^s}{\sqrt{E_x q_x^s (1-q_x^s)}} \approx \frac{\theta_x - E_x q_x^s}{\sqrt{E_x q_x^s}}$$

Since, $q_x^s \approx 0$, we can use the approximation that $1-q_x^s \approx 1$

x	Ex	dx	qx (s)	Ex . qx (s)	dx - Ex.qx
40	50,000	87	0.2053%	102.65	-15.65
41	48,560	84	0.2247%	109.11	-25.11
42	47,190	101	0.2418%	114.11	-13.11
43	44,100	112	0.2602%	114.75	-2.75
44	43,600	123	0.2832%	123.48	-0.48
45	40,400	110	0.3110%	125.64	-15.64
46	37,280	108	0.3438%	128.17	-20.17
47	35,370	122	0.3816%	134.97	-12.97
48	32,100	150	0.4243%	136.20	13.80
49	29,000	139	0.4719%	136.85	2.15
50	26,200	151	0.5244%	137.39	13.61
				1,363.32	**-76.32**

Test statistic = $\dfrac{\sum(\theta_x - E_x q_x^s)}{\sqrt{\sum E_x q_x^s}}$ = -2.07

Under the null hypothesis, this has a Normal distribution. Since this is a two-tailed test, we compare the test statistic, at 95% confidence level: |-2.07| > 1.96. Therefore we reject the null hypothesis and conclude that the standard table rates do not adequately reflect the true mortality rates for the term assurance policyholders.

b) The magnitude of the test statistic is higher than critical value 1.96 at 95% confidence interval which leads to the rejection of null hypothesis. Since the test statistic is negative, this indicates that the mortality rates implied by the standard table are too high compared to the mortality of the term assurance policyholders. This could be due to:

- There is a possibility of a downward bias in the mortality rates of term assurance policyholders of the insurance company. This may be because the insurance company sells policies to mainly high net-worth individuals who might have a better life-style than the average population underlying the standard table. As a result, the mortality rates of the insurance company policyholders tend to be lower than the standard table, on average.
- The variance might be higher than that predicted by the Binomial model, used here. This could be due to presence of duplicate policies. It is possible that some individuals have purchased more than one policy and therefore, the underlying assumption of independence of lives is invalidated.

c) Appropriateness of cumulative deviations for graduated rates

We can recalculate the cumulative deviations using the graduated rates:

x	Ex	dx	qx (g)	Ex . qx (g)	dx - Ex.qx
40	50,000	87	0.1712%	85.58	1.42
41	48,560	84	0.1920%	93.25	-9.25
42	47,190	101	0.2117%	99.89	1.11
43	44,100	112	0.2332%	102.85	9.15
44	43,600	123	0.2597%	113.24	9.76
45	40,400	110	0.2917%	117.85	-7.85
46	37,280	108	0.3296%	122.89	-14.89
47	35,370	122	0.3738%	132.22	-10.22
48	32,100	150	0.4245%	136.27	13.73
49	29,000	139	0.4820%	139.77	-0.77
50	26,200	151	0.5465%	143.19	7.81
				1,287.00	0

It is clear that the graduation has been carried out such that the cumulative deviation is zero, as part of the fitting process. In this case, the cumulative deviations test is invalidated due to the non-random way in which the curve has been fitted to the data. Therefore, we cannot use this test to determine whether the graduated rates are an acceptable reflection of the crude rates from the mortality investigation.

d) The signs test can be used to test for overall bias.
From the individual deviations calculated in part (iii) above, we can see that there are 5 negative and 6 positive deviations.

If the graduated rates do not tend to be higher or lower than the crude rates on average, we would expect roughly half the graduated values to be above the crude rates and half below. It is clear that the signs test is almost perfectly met by the deviations (since there are 5 negative and 6 positive out of 11). Therefore, we can conclude that there is no strong bias present in the graduated rates.

More formally, if deviations above or below have a B(11,0.5) distribution, the probability of 5 or less negative deviations can be calculated as:

$P = 0.5^{11}({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + {}^{11}C_3 + {}^{11}C_4 + {}^{11}C_5) = 0.0488\% * 1024 = 50\%$

Since this is a two-sided test, we will accept the null hypothesis that there is no bias in the graduated rates at any reasonable confidence interval.

From the above, we can draw the following conclusions about the quality of the graduation:

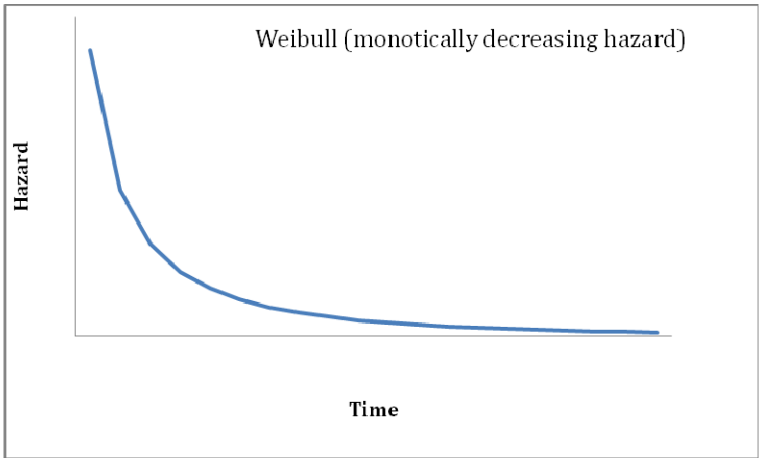
- It is clear that the graduation has been carried out without any bias and such that the cumulative deviations are zero.
- Moreover, the since the graduation is carried out using the standard table, it will automatically result in smooth mortality rates over ages.
- To fully test the goodness of fit, we should carry out other tests of graduation such as chi-square test. This will help in confirming adherence of graduated rates to data and that there are no large off-setting deviations that result in positive test results for cumulative deviations and signs tests.

[16]

Solution 7:

i) Sketch of hazard function and example:

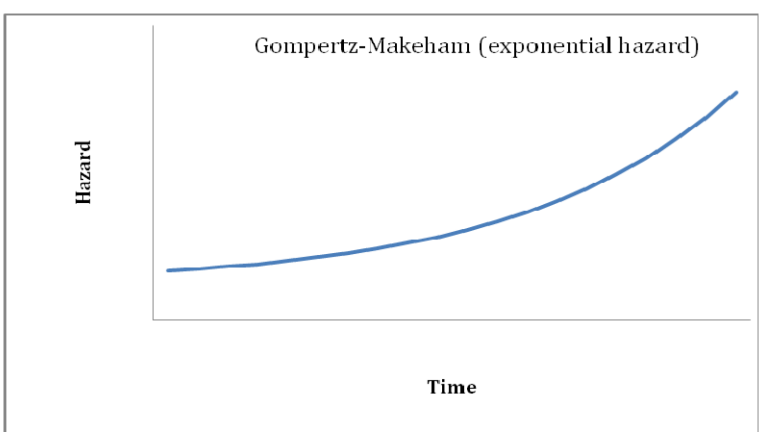
a) Weibull



Example: The decreasing hazard model (decreasing Weibull) could reflect the hazard for patients recovering from major heart surgery. The level of hazard is expected to fall as the time since the operation increases

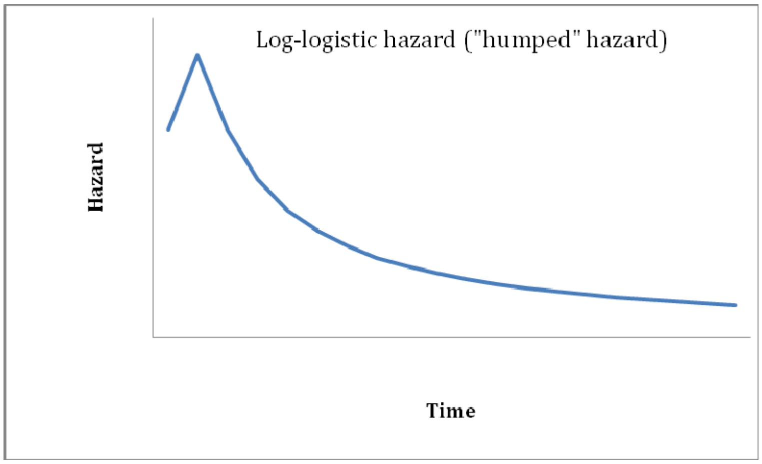
(Note that Weibull model can also be used for a monotonically increasing hazard, though decreasing hazard is more common. Full credit may be given for a graph and a valid example with an increasing hazard)

b) Gompertz-Makenham



Example: The exponentially increasing hazard model (Gompertz-Makeham) could reflect the hazard for leukaemia sufferers who are not responding to treatment. The severity of the condition and the level of hazard increase with the survival time. Over longer time periods, the Gompertz-Makeham model could be suitable for describing the increasing chance of death from natural causes as age increases.

c) Log-logistic



Example: The humped hazard (log-logistic) could reflect a hazard for patients with a disease that is most likely to cause death during the early stages e.g. TB. As the initial condition becomes more severe, the level of hazard increases. But once patients have survived the period of highest risk, the level of hazard decreases.

ii) The force of mortality or hazard rate is assumed constant between integer ages under the Poisson model. Using the Poisson model, the likelihood for each life is proportional to the constant force $\bar{\mu}_{40}$

Thus, likelihood for each life can be written as follows:

Life	1	2	3	4	5
Likelihood	$e^{-\bar{\mu}_{40}}$	$e^{-0.7\bar{\mu}_{40}}$	$e^{-0.6\bar{\mu}_{40}}$	$e^{-0.3\bar{\mu}_{40}} \cdot \bar{\mu}_{40}$	$e^{-0.4\bar{\mu}_{40}} \cdot \bar{\mu}_{40}$

And the total likelihood is the product:

$$L \propto e^{-3\bar{\mu}_{40}} \left(\bar{\mu}_{40}\right)^2$$

iii) Maximising the total likelihood, L:

$$\frac{\partial L}{\partial \bar{\mu}_{40}} = 2\bar{\mu}_{40}.e^{-3\bar{\mu}_{40}} - \left(\bar{\mu}_{40}\right)^2.3e^{-3\bar{\mu}_{40}}$$

Then,

$$\bar{\mu}_{40}.e^{-3\bar{\mu}_{40}}\left[2 - 3.\bar{\mu}_{40}\right] = 0$$

$$\Rightarrow \bar{\mu}_{40} = 2/3$$

This is a maximum because:

$$\frac{\partial^2 L}{\partial \bar{\mu}_{40}^{\,2}} = e^{-3\bar{\mu}_{40}}\left[9\bar{\mu}_{40}^{\,3} - 6\bar{\mu}_{40}^{\,2} - 6\bar{\mu}_{40} + 2\right]$$

Substituting, $\bar{\mu}_{40} = 2/3$

$$\frac{\partial^2 L}{\partial \bar{\mu}_{40}^{\,2}} = -0.27 < 0$$

(Alternatively $\bar{\mu}_{40}$ can be estimated by taking logarithm of the likelihood function and maximizing that, in that case also full marks should be awarded)

Therefore, the maximum likelihood estimate of the hazard rate is 0.667 and the MLE of probability of death can be obtained as:

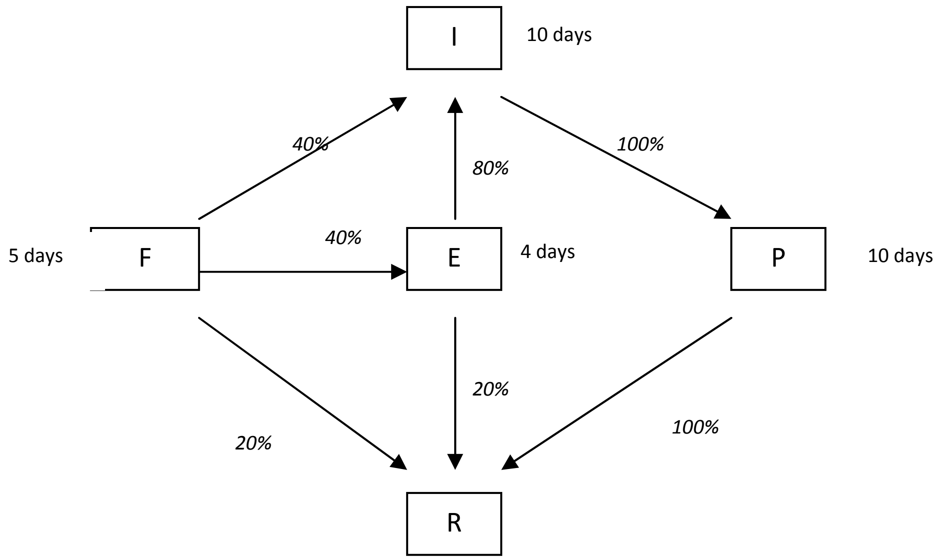
$$\hat{q}_{40} = 1 - e^{-2/3} = 0.4866$$

[14]

Solution 8:

i) Draw the transition diagram and write down the generator matrix for such a Markov process.

Transition diagram:



Per day transition rate:

F=>I: 40%/5 = 8%

F=>E: 40%/5 = 8%

F=>R: 20%/5 = 4%

E=>I: 80%/4= 20%

E=>R: 20%/4= 5%

I=>P: 100%/10= 10%

P=>R: 100%/10= 10%

The generator matrix is set out below:

$$\begin{array}{ccccc} F & I & E & P & R \end{array}$$

$$\begin{bmatrix} -0.20 & 0.08 & 0.08 & 0 & 0.04 \\ 0 & -0.10 & 0 & 0.10 & 0 \\ 0 & 0.20 & -0.25 & 0 & 0.05 \\ 0 & 0 & 0 & -0.10 & 0.10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ii) Further information can be sought from the applicant:

a) Directly by a Patent Officer (40% probability); or

b) By a Technical Expert (Probability of 80% x 40%)

Therefore, further information is sought from the applicant in about 72% of the instances.

iii) Applying the Markov property, the Chapman-Kolmogorov equation is:

$$P_{FF}(t+dt) = P_{FF}(t) \times P_{FF}(dt)$$

Since the probabilities must sum to 1:

$$1 = P_{FF}(dt) + P_{FI}(dt) + P_{FE}(dt) + P_{FP}(dt) + P_{FR}(dt)$$

Applying the definition of transition rates:

$$P_{FI}(dt) \approx 0.08dt + o(dt)$$

$$P_{FE}(dt) \approx 0.08dt + o(dt)$$

$$P_{FR}(dt) \approx 0.04dt + o(dt)$$

$P_{FP}(dt) \approx o(dt)$ because this involves more than one transition in the instantaneous time *dt*.

Substituting:

$$P_{FF}(dt) = 1 - 0.20dt + o(dt)$$

$$\therefore P_{FF}(t+dt) = P_{FF}(t) \times [1 - 0.20dt + o(dt)]$$

$$\therefore P_{FF}(t+dt) = P_{FF}(t) - 0.20P_{FF}(t)dt + o(dt)$$

$$\therefore \frac{P_{FF}(t+dt) - P_{FF}(t)}{dt} = -0.20P_{FF}(t) + \frac{o(dt)}{dt}$$

$$\therefore \frac{dP_{FF}(t)}{dt} = -0.20P_{FF}(t)$$

Using an integrating factor of $e^{0.20t}$ and applying the boundary condition $P_{FF}(0) = 1$ gives:

$$P_{FF}(t) = e^{-0.60t}$$

This is the probability expression that an application is waiting with a Patent Officer for classification at time t.

iv) Using the backward differential equations, we have:

$$P_{FI}(t) = \int_0^t P_{\overline{FF}}(w)\sigma_{FE} P_{EI}(t-w)dw + \int_0^t P_{\overline{FF}}(w)\sigma_{FI} P_{II}(t-w)dw$$

$$\therefore P_{FI}(t) = \int_0^t e^{-0.60w} \times 0.08 \times P_{EI}(t-w)dw + \int_0^t e^{-0.60w} \times 0.08 \times e^{-0.40(t-w)}dw$$

$$\therefore P_{FI}(t) = 0.08\int_0^t e^{-0.60w} P_{EI}(t-w)dw + 0.08e^{-0.40t}\int_0^t e^{-0.40w}\, dw$$

v) To evaluate the expression for $P_{FI}(t)$ we need to derive an expression for $P_{EI}(t-w)$.

$$P_{EI}(t) = \int_0^t P_{\overline{EE}}(w)\sigma_{EI} P_{II}(t-w)dw$$

$$\therefore P_{EI}(t) = \int_0^t e^{-0.25w} \times 0.20 \times e^{-0.40(t-w)}dw$$

$$\therefore P_{EI}(t) = 0.20e^{-0.40t}\int_0^t e^{-0.15w}\, dw$$

$$\therefore P_{EI}(t) = 0.2e^{-0.40t}\left[\frac{e^{-0.15w}}{-0.15}\right]_0^t$$

$$\therefore P_{EI}(t) = \frac{-20}{15}e^{-0.40t}\left[e^{-0.15w}\right]_0^t$$

$$\therefore P_{EI}(t) = \frac{-20}{15}e^{-0.40t}\left[e^{-0.15t} - 1\right]$$

$$\therefore P_{EI}(t) = \frac{4}{3}e^{-0.40t} - \frac{4}{3}e^{-0.55t}$$

Since the process is time-homogenous:

$$\therefore P_{ET}(t-w) = \frac{4}{3}e^{-0.10(t-w)} - \frac{4}{3}e^{-0.25(t-w)}$$

Now, let us consider the expression for $P_{FT}(t)$:

$$P_{FT}(t) = 0.08\int_0^t e^{-0.20w} P_{ET}(t-w)\,dw + 0.08e^{-0.10t}\int_0^t e^{-0.10w}\,dw$$

Consider the second expression in the RHS:

$$0.08e^{-0.10t}\int_0^t e^{-0.10w}\,dw = 0.08e^{-0.10t}\left[\frac{e^{-0.10w}}{-0.10}\right]_0^t$$

$$= \frac{-8}{10}e^{-0.10t}\left[e^{-0.10t} - 1\right]$$

$$= \frac{8}{10}e^{-0.10t} - \frac{8}{10}e^{-0.20t}$$

Now, let us consider the first expression in the RHS:

$$0.08\int_0^t e^{-0.20w} P_{ET}(t-w)\,dw = 0.08\int_0^t e^{-0.20w}\left\{\frac{4}{3}e^{-0.10(t-w)} - \frac{4}{3}e^{-0.25(t-w)}\right\}dw$$

$$= 0.08\int_0^t e^{-0.20w}\,\frac{4}{3}e^{-0.10(t-w)}\,dw - 0.08\int_0^t e^{-0.20w}\,\frac{4}{3}e^{-0.25(t-w)}\,dw$$

$$= \frac{32}{300}e^{-0.10t}\int_0^t e^{-0.10w}\,dw - \frac{32}{300}e^{-0.25t}\int_0^t e^{0.05w}\,dw$$

$$= \frac{32}{300}e^{-0.10t}\left[\frac{e^{-0.10w}}{-0.10}\right]_0^t - \frac{32}{300}e^{-0.25t}\left[\frac{e^{0.05w}}{0.05}\right]_0^t$$

$$= \frac{-32}{30}e^{-0.10t}\left[e^{-0.10t} - 1\right] - \frac{32}{15}e^{-0.25t}\left[e^{0.05t} - 1\right]$$

$$= \frac{-32}{30}e^{-0.20t} + \frac{32}{30}e^{-0.10t} - \frac{32}{15}e^{-0.20t} + \frac{32}{15}e^{-0.25t}$$

$$= \frac{32}{30}e^{-0.10t} - \frac{96}{30}e^{-0.20t} + \frac{32}{15}e^{-0.25t}$$

Adding up the two expressions in the RHS:

$$P_{FI}(t) = \frac{32}{30}e^{-0.10t} - \frac{96}{30}e^{-0.20t} + \frac{32}{15}e^{-0.25t} + \frac{8}{10}e^{-0.10t} - \frac{8}{10}e^{-0.20t}$$

$$P_{FI}(t) = \frac{56}{30}e^{-0.10t} - 4e^{-0.20t} + \frac{32}{15}e^{-0.25t}$$

vi) The closed form solution can be checked by solving for t as zero. By definition, $P_{FI}(0) = 0$.

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