INSTITUTE OF ACTUARIES OF INDIA

CT8 – Financial Economics

November 2010 EXAMINATION

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

A1

(a)

(i) Strong form

The market price is incorporating the publically available information of increase in profits.

(ii) Semi strong form

Either insider information or fundamental analysis leads to high returns.

(iii) Strong form

The fundamental or technical analysis doesn't lead to a better return

(b) It may be possible to forecast some market movements, but the research costs of making the forecasts (or paying a fund manager to do it for you), plus the transaction costs (brokerage, market impact) of executing the deal must be taken into account.

To demonstrate an exploitable opportunity, it must be shown that opportunity is large enough to remain intact even after all these costs are taken into account.

(c) A lot have tests have been done to observe that the change in market value of stocks (observed volatility) can not be justified by the news arriving.

These tests claimed to be evidence of market over-reaction which was not compatible with efficiency.

The early volatility tests were done by Shiller that found strong evidence that the observed level of volatility contradicted the EMH. However, numerous criticisms were subsequently made of Shiller's methodology.

These criticisms covered:

- the choice of terminal value for the stock price
- the use of a constant discount rate
- bias in estimates of the variances because of autocorrelation
- possible non-stationarity of the series, ie the series may have stochastic trends that invalidate the measurements obtained for the variance of the stock price.

Subsequent studies by many authors have attempted to overcome the shortcomings in Shiller's original work there still remains the problem that a model for dividends and distributional assumptions are required.

[16]

A2

(i) When volatility is high, dX_t is strongly upwards; when volatility is low, dX_t is close to zero.

The first of these fits the assumptions, whereas for the second something more negative would be preferable. (It looks as though the process X has more opportunity to increase than to decrease.)

(ii)

(a) Itô (time-independent case): if $dX_t = Y_t\,dt + Z_t\,dB_t$ then

$$df(X_t) = \frac{df}{dx}(Y_t dt + Z_t dB_t) + ½\frac{d^2 f}{dx^2} Z_t^2 dt$$

or simply

$$f'(X_t)dX_t + ½ f''(X_t)(dX_t)^2$$

with an explanation of what is meant by $(dX_t)^2$.
Equally acceptable is the time-dependent version:

$$df(X_t,t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(Y_t dt + Z_t dB_t) + ½\frac{\partial^2 f}{\partial x^2}Z_t^2 dt.$$

(b)

$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\,(dX_t)^2$ [1]

$df(X_t) = f'(X_t)\,[Y_t\, dB_t + Y_t^2\, dt] + \frac{1}{2} f''(X_t)\, Y_t^2\, dt$ [1]

$df(X_t) = f'(X_t)\, Y_t\, dB_t + Y_t^2\,[f'(X_t) + \frac{1}{2} f''(X_t)]\, dt$

In this case,

$f'(X_t) = -2 f(X_t)$
and
$f''(X_t) = 4 f(X_t)$

Therefore
$df(X_t) = -2\, e^{-2Xt}\, V_t\, dB_t$

(c) This is a martingale by the disappearance of the dt term, as $E(dB_t \mid F_t) = 0$.

[9]

A3

(i) CAPM:
$E_i - r = \beta_i (E_m - r)$

Where
Ei is the expected return on security i,
r is the return on the risk-free asset,
Em is the expected return on the market portfolio,
βi is the beta factor of security i defined as cov[Ri ,Rm] / Vm

APT:
$E_i = \alpha + \lambda_1 b_{i,1} + \lambda_2 b_{i,2} + \ldots + \lambda_L b_{i,L}$

Where
Ei is the expected return on security i,
α is constant part of the component of return unique to security i,
λ_k is the additional expected return or risk premium stemming from a unit increase in bi,k – the sensitivity of security i's investment return to index k.

(ii) The main differences in the CAPM and APT models are:

APT identifies several key systematic factors in the returns generating process while the CAPM assumes a single factor
APT recognizes that these key factors can change over time whereas the CAPM's single factor is unchanging.
APT makes fewer assumptions about investor preferences than the CAPM.
The efficiency of the market portfolio is critical for the CAPM.
However, APT does not tell us what factors to include or how many.

[9]

A4

Let x be the proportion invested in the risky asset and return on risky asset and risk free asset be r_x and r respectively. The standard deviation of risky asset is σ_x

$r_a = x\, r_x + (1 - x)\, r$

$\sigma_a^2 = x^2\, \sigma_x^2$

$U(r_a) = x\, r_x + (1 - x)\, r - \frac{1}{2}\, x^2\, \sigma_x^2$

$dU(r_a) / dx = r_x - r - x\, \sigma_x^2$

Or $r_x - r - x\, \sigma_x^2 = 0$

$x = (r_x - r) / \sigma_x^2$

The second order test,

$d^2U(r_a) / dx^2 = -\sigma_x^2$, which is less than zero since σ_x^2 is positive

Therefore x = (12 - 4) / 10 = 80%

The actuarial student should invest 80% in the risky asset and 20% in the risk free asset.

[9]

A5

Variance = $1 / \lambda^2 = 4$

Downside Semi variance = $\int_{-\infty}^{2} (2-x)^2 f(x)dx$

Or $\int_{0}^{2} (4 - 4x + x^2)\lambda e^{-\lambda x} dx$

Or $\int_{0}^{2} 4\lambda e^{-\lambda x} dx - \int_{0}^{2} 4x\lambda e^{-\lambda x} dx + \int_{0}^{2} x^2 \lambda e^{-\lambda x} dx$

$$\int_0^2 4\lambda e^{-\lambda x}\,dx = \left[\frac{4\lambda e^{-\lambda x}}{-\lambda}\right]_0^2 = \left[-4e^{-\lambda x}\right]_0^2 = -4e^{-1} + 4$$

We know that

$$\int_b^a xe^{-\lambda x}dx = \left[\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]_b^a$$

Therefore

$$\int_0^2 4x\lambda e^{-\lambda x}dx = \left[\frac{4\lambda xe^{-\lambda x}}{-\lambda} - \frac{4\lambda e^{-\lambda x}}{\lambda^2}\right]_0^2 = -8e^{-1} - \frac{4e^{-1}}{\lambda} + \frac{4}{\lambda}$$

$$\int_0^2 x^2\lambda e^{-\lambda x}dx = \lambda\left[\frac{x^2e^{-\lambda x}}{-\lambda} - \int\frac{2xe^{-\lambda x}}{-\lambda}dx\right]_0^2$$

$$= \left[-x^2e^{-\lambda x} + 2\left(\frac{xe^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^2$$

$$= -4e^{-1} - \frac{4e^{-1}}{\lambda} - \frac{2e^{-1}}{\lambda^2} + \frac{2}{\lambda^2}$$

Therefore,

$$\text{Downside Semi variance} = -4e^{-1} + 4 + 8e^{-1} + \frac{4e^{-1}}{\lambda} - \frac{4}{\lambda} - 4e^{-1} - \frac{4e^{-1}}{\lambda} - \frac{2e^{-1}}{\lambda^2} + \frac{2}{\lambda^2}$$

$$= \quad -4e^{-1} + 4 + 8e^{-1} + 8e^{-1} - 8 - 4e^{-1} - 8e^{-1} - 8e^{-1} + 8$$

$$= 4 - 8e^{-1} = 1.05696$$

[7]

A6 (i) The updating equation for the force of inflation *I(t)* under Wilkie model is:

I(t) = QMU + QA [I(t-1)-QMU] + QSD.QZ(t)

where:

- *Q* *MU* represents the long-run mean value of *I(t)*
- *QA* represents the "speed" at which *I(t)* reverts to the long-run mean value in each period
  QSD represents the magnitude of the influence of the random innovations in each period
- QZ(t) is a series of independent and identically distributed standard normal variables

(ii) A *cross-sectional property* fixes a time horizon and looks at the distribution over all the simulations.

A *longitudinal* property picks one simulation and looks at a statistic sampled repeatedly from that simulation over a long period of time.

(iii) The parameters of a model should reflect market views of cross sectional distribution. Most statistical properties computed from historical data are effectively longitudinal properties.

In an environment where asset returns are independent across years and also (as for any model) across simulations, cross-sectional and longitudinal quantities coincide. To equate the two is valid under such circumstances.

But the Wilkie model doesn't have the independence property. Hence it is not valid to equate cross-sectional and longitudinal properties.

[10]

A7 Using Greeks we can approximate the change in the price of the option as follows

$$df = \Delta dS + \frac{1}{2}\Gamma dS^2 + \nu d\sigma + \rho dr + \theta dt$$

Substituting the relevant values from the question gives

$$-2.100 = 0.5dS + \frac{1}{2}0.01 * dS^2 + 2*(0.4 - 0.25) + 5*(0.09 - 0.1) - \frac{0.01}{100}*1$$

$$0.005dS^2 + 0.5dS + 2.375 = 0$$

Solving the equation for *dS* gives *-5* or *-95*. Since *-95* is more than 20% movement in the price the value of *dS* should be *-5*.

Therefore the initial value of the stock is 110. **[6]**

A8 (i) The implementation of the strategy involves buying 4 call options. Maximum loss that a trader can incur when buying a call option is limited to the price of the option (ignoring trading expenses). Therefore, the maximum loss on the strategy is equal to the combined price of the 4 call options.

The Black-Scholes formula for the value of a call option on a non-dividend-paying underlying is:

$$f = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where $d_1, d_2 = \dfrac{\left(\ln\frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)T\right)}{\sigma\sqrt{T}}$

We are given that: $S = 100, r = 5\%, \sigma = 10\%, T = 1$

We can evaluate the following

Strike	d_1	$\Phi(d_1)$	d_2	$\Phi(d_2)$	f
100	0.55000	0.70884	0.45000	0.67364	6.80
105	0.06210	0.52476	-0.03790	0.48488	4.05
110	-0.40310	0.34344	-0.50310	0.30745	2.17
120	-1.27322	0.10147	-1.37322	0.08484	0.46

The price paid for implementing the strategy would therefore be Rs 13.49.

(ii) There is no upper limit on the profit that a trader can make on a long call option. It naturally follows that there is no upper limit on the profit under the strategy.

(iii) Payoff on a call option at maturity is given by *max(S(T)-K,0)*. Payoff on the portfolio is as follows

Strike	*Payoff*
100	12.5
105	7.5
110	2.5
120	0
Portfolio	*22.5*

Profit/loss made on strategy is:

Payoff – Cost (including interest)

$$22.5 - 13.49(1 + 10\%) = 7.66$$

(iv) A long call strip gives the holder an increased exposure to a positive movement in the underlying price. A trader may want to adopt this strategy if he holds bullish view on either the stock price or the stock price volatility or both.

[13]

A9 (i) The price of zero coupon bond with ‘n’ year term to maturity is given by

$$B(t, t+n) = e^{\left(-\int_t^{t+n} f(t,u)du\right)}$$

Given the model for forward rate is time homogenous the equation can be reduced to the following form

$$B(\tau) = e^{\left(-\int_0^{\tau} f(u)du\right)}$$

where $\tau = T - t$ *and* $f(\tau) = a + b0.5^{\tau} + c0.5^{2\tau}$

Then

$$\int f(u)du = \int \{a + b0.5^{u} + c0.5^{2u}\}du$$

$$= a\,u + b\left(\frac{0.5^{u}}{\log 0.5}\right) + c\left(\frac{0.5^{2u}}{\log 0.5^{2}}\right)$$

Thus

$$-\log B(\tau) = a\tau + b\frac{0.5^{\tau} - 1}{\log 0.5} + c\frac{0.5^{2\tau} - 1}{2\log 0.5}$$

Using the price of zero coupon bonds we can form a system of three equation in three variables.

Students can adopt different approaches to solve this system. One such approach is described below. Provided the steps are clearly defined marks should be awarded for correct solution irrespective of the approach adopted.

The system of equations can be presented in the following matrix equation

$AX = B$ and the solution is given by $X = A^{-1}B$

where

$$A = \begin{Bmatrix} 0.25 & 0.229\,538 & 0.211\,278 \\ 2 & 1.082\,021 & 0.676\,268 \\ 10 & 1.441\,286 & 0.721\,347 \end{Bmatrix}, X = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}, B = \begin{Bmatrix} 0.018\,088 \\ 0.247\,338 \\ 1.107\,921 \end{Bmatrix}$$

$$A^{-1} = \begin{Bmatrix} 0.384\,917 & -0.275\,412 & 0.145\,460 \\ 10.545\,741 & 3.830\,680 & 0.502\,494 \\ 15.734\,828 & -3.835\,875 & 0.373\,804 \end{Bmatrix}$$

Thus

$$X = \begin{Bmatrix} 10\% \\ 20\% \\ -25\% \end{Bmatrix}$$

Therefore the instantaneous forward rate *f(t,T)* is given by

$f(t,T) = 10\% + 20\% * 0.5^{(T-t)} - 15\% * 0.5^{2(T-t)}$

[Total 8]

(ii) From (i) we have

$$-\log B(t) = at + b\frac{0.5^t - 1}{\log 0.5} + c\frac{0.5^{2t} - 1}{2\log 0.5}$$

Therefore

$$-\log B(3) = 10\% * 3 + 20\% * \frac{0.5^3 - 1}{\log 0.5} - 25\%\frac{0.5^6 - 1}{2\log 0.5}$$

$B(3) = 0.6873219 \text{ or } 68.73\%$

(iii) The price of a zero-coupon bond maturing at time T, which pays 1 if default has not yet occurred and δ if default has occurred and for which the credit rating of the underlying corporate entity is i , is given by

$$B(t,T,X(t) = i) = P(t,T)\{1 - (1 - \delta)P_Q[X(T) = n \mid X(t) = i]\}$$

where

P(t,T): market price at time *t* of a *default-free* zero-coupon bond that pays 1 unit at time *T*

$P_tQ[X(T) = n \mid X(t) = i]$: probability that the bond is in state of default at time T given that the current state is *i*

We want to estimate

$B(0,3, X(0) = \text{BBB})$

We know that

P(0, 3) = 68.73% from (ii)

δ = 30%

We can estimate $P_tQ[X(3) = \text{D} \mid X(0) = BBB]$ using the following formula

$P_tQ[X(3) = D \mid X(t) = BBB] = S\Pi^3 \,|_{1,4}$

where

$S = (0 \quad 1 \quad 0 \quad 0)$

and

$$\Pi = \begin{pmatrix} 0.5 & 0.3 & 0.2 & 0 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \Pi^2 = \begin{pmatrix} 0.36 & 0.34 & 0.23 & 0.07 \\ 0.31 & 0.36 & 0.16 & 0.17 \\ 0.16 & 0.23 & 0.29 & 0.32 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \Pi^3 = \begin{pmatrix} 0.305 & 0.324 & 0.221 & 0.150 \\ 0.279 & 0.305 & 0.178 & 0.238 \\ 0.178 & 0.221 & 0.200 & 0.401 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$S\Pi^3 \,\big|_{1,4} = 0.238$

$B(0,3, X(0) = \text{BBB}) = 0.6873\{1 - (1 - 0.3)0.238\}$

$B(0,3, X(0) = \text{BBB}) = 57.28\%$

[16]

A10 (i) Let V(t), St and Bt be the value at time t of the portfolio, the stock and the cash bond respectively. We can depict the value of the portfolio as follows:

$V(t) = \varphi_t S_t + \psi_t B_t$

Further, let $dV(t)$, dS_t and dB_t be the instantaneous change in the value of the portfolio, the stock and the cash bond respectively.

The portfolio strategy will then be described as self-financing if dV (t) is equal to $\varphi_t dS_t + \psi_t dB_t$: that is, at t+dt there is no inflow or outflow of money necessary to make the value of the portfolio back up to V(t +dt)

(ii) A replicating strategy is a self-financing strategy (φ_t, ψ_t), defined for $0<t<U$, such that:

$V(U)= \varphi_U S_U + \psi_U B_U =X$

So, for an initial investment of V(0) at time 0, if we follow the self-financing portfolio strategy (φ_t, ψ_t) we will be able to reproduce the derivative payment without risk.

Since holding the derivative generates the same payoff as holding the portfolio at time U and that there are no intermediate cashflows, by principle of no arbitrage the two should have the same price at time 0. Therefore we have

Price of derivative $= V(0) = \varphi_0 S_0 + \psi_0 B_0$

[5]

[Total Marks 100]

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