Institute of Actuaries of India

Subject CT1 – Financial Mathematics

November 2010 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1 a) i) Accumulated value after 3 years = (1000 + 1000 *0.06 * 3)
= 1180

Accumulated value after 5 years = (1180 + 1180*0.07*2)
= 1345.20

ii) Accumulated value after 5 years = $1000(1+\frac{.05}{12})^{30}(1+.02)^{4}\exp(\delta*1.5)$ where $\delta=0.08$
= 1382.53

[4]

b) Given $d^{(4)} = 0.08$

i) To Find $i^{(2)}$
$d^{(4)} = 4(1 - v^{1/4})$
$v = (1- 0.08/4)^{4} = 0.92236816$
$i^{(2)} = 2((1+i)^{½} -1) = 2(0.92236816\text{^}-(1/2)-1) = 0.082465639$
= 8.2465%

ii) To Find $d^{(12)}$
$d^{(12)} = 12 (1 - v^{1/12})$
= 12(1- 0.92236816^(1/12))
= 0.080539339
= 8.0539%

iii) To find $i^{(½)}$
$i^{(½)}$= ½*((1+i)^2-1)
= ½*(v^-2 – 1)
= ½*(0.92236816^(-2) – 1) value of 'v' from part (i) above
= 0.087707724
= 8.7708%

[4]
[Total 8]

2 a) i) **Real Rate of Return** : Given any series of monetary payments accumulated over a period, <u>a real rate of interest</u> is that rate which will have earned so as to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effects of inflation.

Real rate of interest = i' = (i-e)/(1+e)
Where i = money rate of interest i.e., effective rate of interest with no inflation adjustment
e = rate of inflation

ii) **Time Weighted Rate of Return** is the product of growth factors between consecutive cash flows. It is given by:

$$(1+i)^{T} = \frac{F_{t1-}}{F_0+C_0} \times \frac{F_{t2-}}{F_{t1-}+C_{t1}} \times \frac{F_{t3-}}{F_{t2-}+C_{t2}} \times \ldots\ldots\ldots \frac{F_T}{F_{tn-}+C_{tn}}.$$

Where i = TWRR
F_0 = Value of a fund at time t= 0

C_{tk} = Net cash flow at times t_k, k= 1,2,3,4....n
F_T = Fund value at time T >= t_n
C_0 = Cash flow at time t=0
F_{tk-} = Value of fund just before the cash flow due at time t_k

[4]

b) Base index = 115

Coupon at end of year 1 = 0.05 * 400,000 * 1.23/1.15 = 21,391.30
Coupon at end of year 2 = 0.05 * 400,000 * 1.28/1.15 = 22,260.87
Coupon at end of year 3 = 0.05 * 400,000 * 1.33/1.15 = 23,130.43
Coupon at end of year 4 = 0.05 * 400,000 * 1.41/1.15 = 24,521.74

Coupons are reinvested at 4% p.a. effective in a bank.
Accumulation of
coupons = $21391.3(1.04)^3+22260.87(1.04)^2+23130.43(1.04)+24521.74$
= 96,717.05 ---------------------------- (A)

Redemption value at maturity = 400,000*1.41/1.15
= 490,434.78 ----------------------------- (B)

Total Proceeds = (A) + (B)
= 96,717.05 + 490,434.78
= 587,151.83

Current Cost of car =450,000
Inflation of cost over next 4 years is @ 5% p.a.
Hence expected cost of car after 4 years= $450,000(1.05)^4$
= 546,977.81

Since the expected cost of car < Total investment proceeds,

we may conclude that the investment proceeds would be sufficient to buy the car.

<u>Alternative method for maturity proceeds (Working in 100 nominal)</u>

Coupon at end of year 1 = 0.05 * 100* 1.23/1.15 = 5.34783
Coupon at end of year 2 = 0.05 * 100* 1.28/1.15 = 5.56522
Coupon at end of year 3 = 0.05 * 100* 1.33/1.15 = 5.78261
Coupon at end of year 4 = 0.05 * 100* 1.41/1.15 = 6.13043
Redemption at end of year 4=100*1.41/1.15 = 122.6087

Accum of coupons @4% p.a.=
= $5.34783(1.04)^3+5.56522(1.04)^2+5.78261(1.04)+6.13043+122.6087$
= 146.78797

Total maturity proceeds = 146.78797*400,000/100 = 587151.87

[5]
[Total 9]

3 a) Let monthly installment = X

Effective rate of interest p.a. = $1.04^2 - 1 = 8.16\%$

Then

$12X\ddot{a}_{\overline{20}|}^{(12)} = v^{20}(10{,}000*12)a_{\overline{1}|}^{(12)}(1+1.03v+1.03^2v^2+1.03^3v^3+\ldots+1.03^{19}v^{19})$ @8.16%

$121.512538X = 120{,}000\ v^{20}a_{\overline{1}|}^{(12)}(1 - 1.03^{20}v^{20})/(1-1.03v)$ @ 8.16% (sum of a GP)

$= 313308.77$

<u>**X = 2578.407**</u>

[6]

b) Effective rate of interest p.a. = $(1+0.09/4)^4 - 1 = 9.3083\%$

PV of the 3-year annuity = $6{,}500[\bar{a}_{\overline{1}|} + v\ddot{a}_{\overline{1}|}^{(12)} + v^2\ddot{a}_{\overline{1}|}^{(2)}]$ @ 9.3083%

$= 6500[\ 0.956790 + 0.878564 + 0.818724]$

$= 6219.135 + 5710.666 + 5321.706$

$= 17251.50$

<u>**Present value of these payments = 17,251.50**</u>

[8]

[Total 14]

4 a) Let Y be the Amount of loan . Then,

$Y = (165-5)v + (165-5*2)v^2 + (165-5*3)v^3 + \ldots + (165-5*15)v^{15}$ @ 8%

$= 165a_{\overline{15}|} - 5\ (Ia)_{\overline{15}|}$ @ 8%

$= 165(8.559479) - 5(56.445143)$

$= 1130.0883$

[3]

b) 3^{rd} installment = 165-15 = 150

Interest content of 1^{st} installment = 1130.0883*0.08 = 90.407
Capital content of 1^{st} installment = 160 – 90.407 = 69.593

Interest content of 2^{nd} installment = (1130.0883 – 69.593)*0.08
= 1060.495*0.08= 84.840
Capital content of 2^{nd} installment = 155 – 84.84 = 70.16

Interest content of 3^{rd} installment = (1060.495 – 70.16)*0.08 =79.227
Capital content of 3^{rd} installment = 150 – 79.227 = 70.773

Alternative method I for loan o/s after 2 installments (Prospective):

Loan o/s after
2 installments $=(155-5)v+(155-5*2)v^2+(155-5*3)v^3+ \ldots +(155-5*13)v^{13}$ @8%
$= 155a_{\overline{13}|} - 5\,(Ia)_{\overline{13}|}$ @ 8%
= 155(7.903776)-5(46.95006)
= 990.335

Interest content/Capital content of 3^{rd} installment : Same as above

Alternative method II for loan o/s after 2 installments(Retrospective) :

Loan o/s after
2 installments = $1130.0883(1.08)^2$ - 160*1.08 – 155
= 990.335
Interest content/Capital content of 3^{rd} installment : Same as above

[3]

c) 13^{th} installment = (165-5*13)= 100

Capital o/s after 12 installments = $100v + 95v^2 + 90v^3$ @ 8%
= 245.485

Interest content of 13^{th} installment = 245.485*0.08 = 19.6388

Capital content of 13^{th} installment = 100 – 19.6388 = 80.3612

[3]

[Total 9]

5 a) i) **Index-linked Security** : It is a security that provides a series of regular interest payments and a final redemption(Maturity) payment where the actual cash amount of interest payments and of the final redemption payment are linked to an index which reflects the effect of inflation.

Cash flows : The investor has an initial negative cash flow (price paid for purchase of the security), a series of unknown positive cash flows on a regular set of specified future dates and a single larger unknown positive cash flow on a specified future date. The positive cash flows at any time will relate to an inflation index.

ii) **Call Deposits** : A cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider.

Cash flows: The investor has an initial negative cash flow. The interest payments will depend on the current interest rate and will be added to the initial investment. The money can be disinvested at any time at investor's choice.

iii) Equity : Equity shares are securities that are held by owners of an organization. Shareholders are entitled to a share in the company's profits in proportion of the number of shares owned.

Cash flows : The investor has an initial negative cash flow, a series of unknown positive cash flows which will depend on the profits earned by company. On disinvesting at any time at investor's choice, there will be another positive cashflow which may be more or less than the initial negative cashflow depending on the financial performance of the company and the market conditions.

[6]

b) Price of share =

$= 32v^{(2/12)}*(1+(1.04.v)^{(1/2)}+1.04.v+(1.04.v)^{(3/2)}+\ldots\ldots)*0.8$ @ 9%

$= 25.6(0.985740)\dfrac{1}{(1-(1.04.v)^{(1/2)})}$ (Sum of a GP)

= 25.23494/0.023205

= 1087.478

~ 1087.48

[4]

c) Capital gains tax on sale at end of 2 years per share= (1114-989)*0.1 = 12.5
Net amount received on sale of 50 shares = 50*(1114-12.5)= 55,075

Capital gains tax on sale at end of 7 years per share = (2340-989)*0.1 = 135.1
Net amount received on sale of 50 shares = 50*(2340-135.1)= 110,245

The required equation of value is:

$$98900 = 55075v^2+110245v^7+\frac{32*50*0.8}{1.04^{1/2}}\{(k+k^2+k^3+\ldots k^{14})+(k+k^2+k^3+k^4)\}$$

Where $k = +(1.04.v)^{(1/2)}$

$$= 55075v^2+110245v^7+1255.143\{k\frac{(1-k^{14})}{1-k}+k\frac{(1-k^4)}{1-k}\}$$

We can start with 10% because the price paid by Eva is lesser than the 9% price of part (b).

At 10% RHS = 121102.5484
At 15% RHS = 99825.98552
At 16% RHS = 96277.96045

Interpolating between 15% and 16% we get **i=15.261%**

Interpolation& calculation at different 3 marks
CGT 1 mark
Equation and simplification 4 marks

[8]

[Total 18]

6 a) PV of outgo $= 150 + 20\ (v^{1/12} + v^{2/12} + \ldots + v^{6/12}) + 3\ v^{1/2}\ a^{(12)}_{\overline{19.5}|}$ @ 8%

$= 150 + 19.75\ (v^{1/12} + v^{2/12} + \ldots + v^{6/12}) + 3\ a^{(12)}_{\overline{20}|}$

$= 296.3958$

PV of Income $= 3*12\ v^{1/2}\ \ddot{a}^{(12)}_{\overline{19.5}|} + 500\ v^{20}$

$= 350.8735 + 107.275$

$= 458.1485$

Net Present Value = PV(Income)-PV(outgo)

= 458.1485 - 296.3958

= **161.753**

[7]

b) DPP is the earliest time 't' such that

$- (150 + 20\ (v^{1/12} + v^{2/12} + \ldots + v^{6/12}) + 3v^{½}\mathbf{a}^{(12)}_{\overline{t-½}|}) + 36\ v^{½}\ddot{\mathbf{a}}^{(12)}_{\overline{t-½}|} >= 0$

$- 3v^{½}\mathbf{a}^{(12)}_{\overline{t-½}|} + 36\ v^{5/12}\ \mathbf{a}^{(12)}_{\overline{t-½}|} >= 267.34341$

$+ \mathbf{a}^{(12)}_{\overline{t-½}|}\ [\ -3v^{½} + 36\ v^{5/12}\ ] >= 267.34341$

$a^{(12)}_{\overline{t}|} >= (267.34341 / 31.97715) = 8.3604$

$=> v^{t-½} >= 0.354503$

taking log on both sides

$=> t - ½ >= 13.47484$

$=> t >= 13.97484$ years

[5]

[Total 12]

7 a) Present value of the liabilities @ 7% per annum effective=

$= 85000a_{\overline{14}|} + 300,000\ v^{11}$ @ 7%

= 85000(8.745468)+ 142527.8389

= 885892.62

= **885893**

b) DMT of liabilities @ 7% =

$$= \frac{85000(Ia)_{\overline{14}|} + 300000(11\ v^{11})}{885,893} \text{ @ } 7\%$$

= (85000(56.117277)+ 1567806.228)/885893

= 6337774.749/885893

= **7.1541 years**

c) Let P & Q be the nominal amounts of Bond P & Bond Q respectively to be invested today.

If PV(Assets) = PV(Liabilities), then
$P(0.09)a_{\overline{9}|} + Pv^{9} + Q(0.055)a_{\overline{24}|} + Qv^{24} = 885893$
$P(0.586371+0.543934)+Q(0.630813+0.197147) = 885893$
$1.130305 P + 0.82796 Q = 885893$ ----------------------------------(I)

DMT(Assets)=DMT(Liabilities) implies

$$\frac{P(0.09)(Ia)_{\overline{9}|} + 9Pv^{9} + Q(0.055)(Ia)_{\overline{24}|} + 24Qv^{24}}{885893} = 7.1541$$

$7.564411 P + 10.656330 Q = 6337775$
$\Rightarrow Q = (6337775 – 7.564411 P)/10.656330$ ---------------------(II)

Substitute this in equation (I)

$1.130305 P + 0.82796 (6337775 – 7.564411 P)/10.656330 = 885,893$
$P(0.542576) + 492423 = 885893$
$P = 393,469.39/0.542576 = 725,188$
P = 725,188

Substitute this value of P in (II)

- ⇨ $Q = (6337775 – 7.564411(725188))/10.656330$
- ⇨ **Q = 79967**

d) It appears that the asset payments are more spread out than the liability payments. The third condition for immunization is that that convexity of the assets is greater than that of the liabilities, or that the asset times are more spread around the discounted mean term than the liability times. From observation is appears likely that this condition is met.

[Total 15]

8 a) The mean accumulated amount after 10 years of a single investment of Rs. 1000 at time 0 is $1000*(1+0.0675)^{10}$, where 0.0675 is the mean rate of interest in any year.

- ⇨ The expected value of accumulated amount = $1000*(1+0.0675)^{10}$ = **1921.67**

$$S_{10} = \prod_{t=1}^{10} (1+ i_t)$$

Given that : $(1+i_t) \sim \log N(\mu, \sigma^2)$

- ⇨ $S_{10} \sim \log N(10\mu, 10\sigma^2)$

Given that $E(1+i_t) = 1.0675 = \exp[\mu + \sigma^2/2]$

$Var(1+i_t) = \exp(2\mu + \sigma^2) [\exp(\sigma^2) – 1]$

$= 0.05^2$

⇨ $(1.0675)^2 [\exp(\sigma^2) - 1] = 0.05^2$

⇨ $\sigma^2 = \ln(1.0021938)$
$= 0.0021914$

$\exp[\mu + \sigma^2/2] = 1.0675$
⇨ $\mu + \sigma^2/2 = \ln(1.0675)$
⇨ $\mu = 0.064224$

⇨ $S_{10} \sim \log N(0.64224, 0.021914)$

⇨ $Var(S_{10}) = \exp[2(0.64224) + 0.021914] * [\exp(0.021914) - 1]$
$= 0.081818$

The Standard Deviation of accumulated amount = $1000 * \sqrt{(0.081818)}$
= **286.04**

[8]

b) The accumulated value at the end of 4 years = $1000 * (1.07)^4$
= 1310.796

The expected value of accumulated value at the end of 10 years
$= 1310.796 [(1.04)^6 * 0.1 + (1.05)^6 * 0.3 + (1.08)^6 * 0.3]$
= 1310.796 [1.44236]
= **1890.65**

$E[X^2] = (1310.796)^2 [0.1*(1.04)^{12} + 0.3 (1.05^{12} + 1.065^{12} + 1.08^{12})]$

= 1718196.64 [2.09304]
= 3596254.342

Therefore, Variation of accumulated value $= E[X^2] - (E[X])^2$
$= 3596254.34 - (1890.65)^2$
= 21696.92

The Standard Deviation of accumulated amount = $\sqrt{21696.92}$ = **147.30**

[7]
[Total 15]