Institute of Actuaries of India

Subject ST6 – Finance and Investment B

May 2015 Examinations

INDICATIVE SOLUTIONS

Solution 1 :

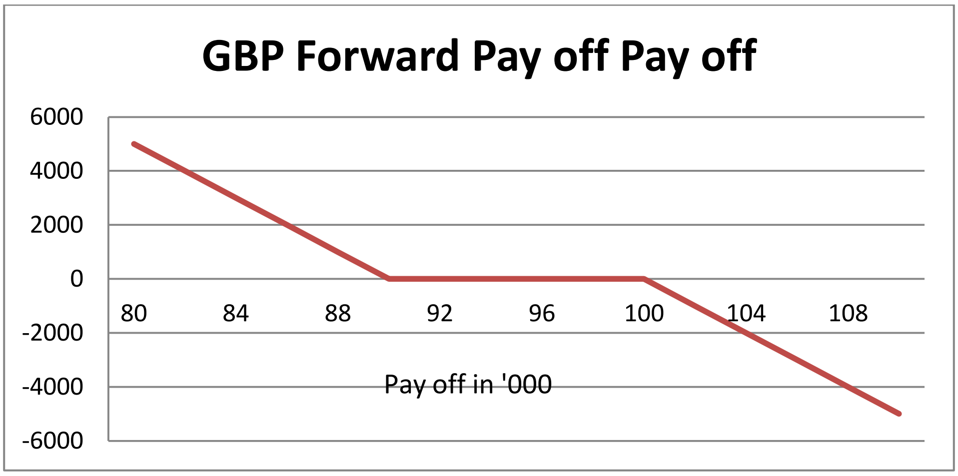
i) The payoff is dependent on two limits SGB1/SUS1 and SGB2/SUS2 if the spot rates at t=90/180 are between these range then there is no payoff thus payoff is

P(GBP_{180})	
-500K (GBP_{180}-SGB2)	If GBP_{180}>100
0	If GBP_{180} is between 90 -100
500K (SGB2-GBP_{180})	If GBP_{180} is below 90

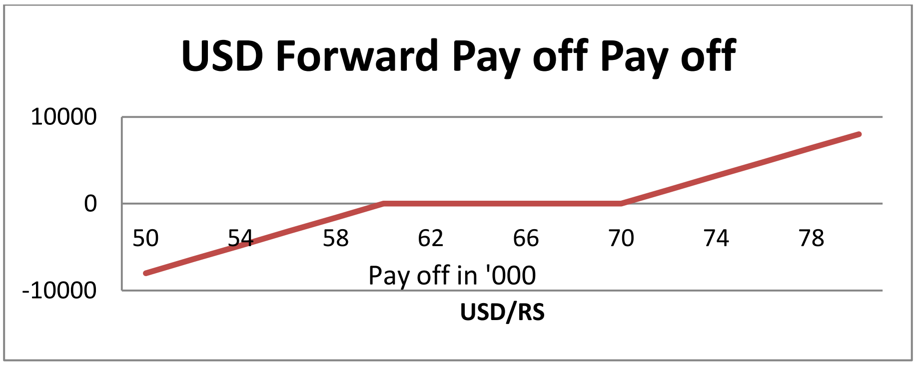
P(USD_{90}	
Similarly for US currency	
800K (USD_{90}-SUS2)	If USD_{90}>60
0	If USD_{90} is between 60 -70
- 800K (SUS2-USD_{90})	If USD_{90} is below 60

Payoff Diagrams

GBP Payoff



USD payoff



[4]

ii) The above payoff is similar to combination of short call and log put for GBP and long call and short put for USD. This can be shown easily by comparing the cashflow for three ranges of spot rates. **[4]**

iii) FX options can be priced using the following modification of the BS equation

$$C(X,t) = e^{-r_f(T-t)} X_t \Phi(d_1) - e^{-r_d(T-t)} K \Phi(d_2)$$

$$\text{where } d_1 = \frac{\log\left(\frac{X_t}{K}\right) + (r_d - r_f + \sigma_x^2/2)(T-t)}{\sigma_x\sqrt{T-t}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

r_d and r_f denote the domestic and foreign risk-free rates

The calculation for the put options are shown in the table below

	GBP 90 Put long	GBP 100 call short	USD 60 put short	USD 70 call long
Spot	90	90	60	60
Strike	90	100	60	70
Duration(yr)	0.42	0.42	0.17	0.17
Volatility	15%	15%	25%	25%
rate-rs	7.50%	7.50%	7.50%	7.50%
rate-fx	0.50%	0.50%	0.50%	0.50%
d1	0.37	-0.72	0.17	-1.34
d2	0.27	-0.81	0.07	-1.44
Price	2.20	-1.11	-2.04	0.24
Nominal (K)	500.00	500.00	800.00	800.00
Total	1101.65	-554.89	-1635.08	194.22
Total	-894.11			

[12]

iv) The main risks are

- Counterparty- the bank may default on the payments in that scenario the company would be exposed to the volatility in the spot rate at the date of the transaction.
- Cash flow uncertainty- There could be business issues which may further delay one or both of the cash flow. In scenario the FX would still have to be settled and any additional payment required as per the contract has to be made. This may put cash flow pressure on the firm.

Counterparty risk can be mitigated by having collaterals against the forward.

[4]

[24 marks]

Solution 2 :

i) Passive hedging – for passive hedging the company would buy a 6 month put option on BSE to cover the possible loss in the value of assets if the BSE fall by 5%.

Active hedging – this is achieved by investing a portion of portfolio to risk free security to achieve portfolio delta similar having put options.

The active hedging is expensive but flexible. There would be frequent transaction cost of buying and selling the portfolio. **[5]**

ii)

Input Data	
BSE now	8600
95% level put option	8170
Time	0.5
Volatility	30.00%
Dividend	2%
Risk free rate	6%
Calculation	
Forward value	8773.7315
d1	0.4421
d2	0.2300
Norm(-d1)	0.3292
Norm(-d2)	0.4090
Value of Put	**440.2137**

The cost of hedging at is (440.2/8600*1000)= 51Cr (4pprox..).

Delta of the put option is

$$e^{-q(T-t)}\,(N(d_1) - 1)$$

$$-e^{-qT}\Phi(-d_1)$$

Delta is 33.25% hence company has to sell 33.25% of the portfolio initially and invest in the risk free bonds for the purpose of the active hedging. **[6]**

[11 Marks]

Solution 3 :

i)

Cashflow	**A**	1	Default amount	Net principle	Recovery	interest	Default adj
Year1	Cashflow	A Default rate	A	A	A	A	A
0	1000	0					-1000
1	50	2%	20	980	10	49	59
2	50	5%	49	951	15	48	62
3	1050	8%	83	917	17	46	980
							3.457%

IRR for par bond is same as interest which is 5% thus spread of 1.55% over risk rate which is .45% higher than he market spread.

[10]

ii) Some of the reasons are:

- The market value would have premium of illiquidity and other sources of volatility not related to default or transitions.
- Investor is not risk averse and require additional premium over historically estimated risk free rates for adverse scenarios.
- The model only consider best estimate scenario, the give default and transition probabilities may not materialise in future. The expected default could be much higher than expected.
- The model also assumes a portfolio of the bonds if the company only decides to invest in couple of bonds without diversification there is a possibility higher loss.

[4]

[14 Marks]

Solution 4 :

i) Value of fixed leg = $c\{v + v^2 + v^3 + v^4 + v^5\}$ = $[cv/(1-v)]\ [1 - v^5]$

$v = [1/(1+i)] \Rightarrow (1/v) - 1 = i$

Thus, the value of fixed leg = $[\{c\ (1 - v^5)\}\ /i]$

The value of floating leg = $L_0v + L_1v^2 + \ldots.. + L_4v^5$ where $v^{k-1} = (1 + L_{k-1})v^k$; k = 1, …., 5

i.e. $L_{k-1}v^k = v^{k-1} - v^k$.

Thus, value of floating leg = $1 - v^5$; it follows that $[\{c\ (1 - v^5)\}\ /i] = 1 - v^5 \Rightarrow c_1 = i$

[3]

ii) The semiannual interest rate j is given by $(1 + j/2)^2 = 1 + I \Rightarrow j = 2\{(1 + i)^{½} - 1\}$
Using the same argument as above,

$$c_2 = (j/2) = \{(1 + i)^{½} - 1\}$$

[1]

iii) The value of the floating leg is now given by

$$FL = L_1v^1 + L_2v^2 + L_3v^3 + L_4v^4 + L_5v^5$$

Where $v_k = (1 + L_k)v^{k+1}$
i.e. $[L_kv^k]v = v^k - v^{k+1} \Rightarrow L_kv^k = v^{k-1} - v^k$ as before.
Thus, $c_3 = c_1$

[3]

iv) The value of the floating leg is now given by

$$Fl = s(5)_0v^1 + s(5)_1v^2 + \dots + s(5)_4v_5$$

where $s(5)_t$ = the zero value 5 year swap coupon at time t.

Using the argument in (a) above, the coupon on a forward slant 5 year swap commencing k periods in the future is given by

$$Fxd = v^k . s(5)_k . [v + \dots + v^5] = [v^k s(5)_k (1 - v^5)]/ i$$

$\Rightarrow$ Flt = $v^k (1 - v^5)$.
Thus, $s(5)_k = I = c_1$; it follows that $c_4 = c_1$.

[3]

v) The primary impact of the upwards sloping yield curve is that forward rates increase over time in relation to the curve.

Thus, c_3 new > c_3 old; and c_4 new > c_4 old

[2]
[12 Marks]

<u>Solution 5 :</u>

i) Given a probability measure **P** and a history (filtration) of past events $\{F_t , t \leq s\}$, then the stochastic process $\{X_t , t \geq 0\}$ is a martingale if:

$$E_P[X_t \mid F_s] = X_s \text{ for any } t \geq s.$$

In other words, the expected future value of the stochastic process X_t is its current value, i.e. it is driftless.

[2]

ii) The variable ϕ represents the value of f relative to the price of g. It can be thought of as *measuring* the price of f in units of g.

The security g is said to be the numeraire asset, i.e. the asset by reference to which the worth of other assets (e.g. f, an arbitrary function) is measured.

[4]

iii) We have:

$$\frac{\mu_f - r}{\sigma_f} = \lambda = \frac{\mu_g - r}{\sigma_g}$$

which implies

$$\mu_f = \sigma_f\lambda + r \text{ and } \mu_g = \sigma_g\lambda + r$$

Substituting λ = σ_g gives

$$\mu_f = \sigma_f\sigma_g + r \text{ and } \mu_f = \sigma_g^2 + r$$

(1/2 mark each for getting the equation right)

Substituting these equations into the SDE s for *f* and *g* gives

$$df = (\sigma_f\sigma_g + r)fdt + \sigma_f fdz \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$df = (\sigma_g^2 + r)gdt + \sigma_g gdz \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Using Ito s Lemma on (1) and (2) for the functions ln *f* and ln *g* respectively, we get:

$$dlnf = \left(\sigma_f\sigma_g + r - \frac{\sigma_f^2}{2}\right)fdt + \sigma_f dz$$

$$df = \left(\frac{\sigma_g^2}{2} + r\right)gdt + \sigma_g dz$$

Note that,

$$\frac{d}{df}(lnf) = \frac{1}{f} \text{ and } \frac{d^2}{df^2}(lnf) = -\frac{1}{f^2}$$

So that, taking differences,

$$d(lnf - lng) = \left(\sigma_g\sigma_f - \frac{\sigma_f^2}{2} - \frac{\sigma_g^2}{2}\right)dt + (\sigma_f - \sigma_g)dz$$

Equivalently,

$$d\left(ln\frac{f}{g}\right) = -\frac{(\sigma_f - \sigma_g)^2}{2}dt + (\sigma_f - \sigma_g)dz$$

Using Ito s Lemma in reverse (comparing to formulae (1) and (2) above), we can write down the process for *f* / *g* that gives such a result for ln(*f* / *g*)

$$d\left(\frac{f}{g}\right) = (\sigma_f - \sigma_g)\frac{f}{g}dz$$

This is a driftless process, and is hence a Martingale.

[4]

iv) A world which consists of a security (or stochastic process) *g* whose volatility σ_g is equal to the market price of risk, then the world is said to be forward risk neutral with respect to *g*.

If *f* is any other security, then in the world that is forward risk neutral with respect to g, *f* / *g* is a Martingale. It follows that the current value of *f* / *g*, namely f_0/g_0 is equal to the expected value at time zero of all future values $E_g[\ f_t\ /\ g_t]$, where expectations are carried out using the probability density function underlying g.

[5]

[15 Marks]

Solution 6 :

i) The Cameron-Martin-Girsanov Theorem (CMG)

If W_t is a **P**-Brownian motion and γ_t is an F-previsible process (i.e. a variable known at time t based on past history filtration F) satisfying the boundedness condition

$$E_P\left[exp\left(\frac{1}{2}\int_0^T \gamma_t^2 dt\right)\right] < \infty$$

then there exists a measure **Q** such that **Q** is equivalent to **P**

$$\frac{dQ}{dP} = exp\left(-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt\right)$$

$$\widetilde{W}_t = W_t + \int_o^T \gamma_s ds \text{ is } \mathbf{Q} \text{ - Brownian Motion}$$

[3]

ii) Martingale Representation Theorem

If M_t is a **Q**-martingale process whose volatility σ_t satisfies the additional condition that it is (with probability one) always non-zero.

Then if N_t is any other **Q**-martingale, there exists an F-previsible process φ such that N_t can be written as

$$N_t = N_0 + \int_o^t \varphi_s dM_s$$

[2]

iii) Significance of Cameron Martin Girsanov (CMG) theorem and Martingale Representation Theorem (MRT) for valuing derivatives

Derivatives can be valued by finding a portfolio of cash and stock which replicates the derivative. The value of the derivative must be equal to the value of the replicating portfolio otherwise arbitrage opportunities would exist.

The approach for finding the replicating portfolio involves finding a probability measure **Q** that makes the underlying discounted stock price process $Z_t = B_t^{-1} S_t$ a martingale, where S_t is the stock price and B_t is the riskfree zero coupon bond price at time t. This means that the expectation of all future values of Z_t is equal to its value at time 0, i.e. S_0.

Converting a stochastic process into a martingale involves finding measure **Q** under which it is driftless. The CMG theorem confirms the existence of such a measure. It gives us a powerful tool for controlling the drift of most stochastic processes that are encountered in practice.

If the process Z above is a martingale, then the process $E_t = E_Q[B_t^{-1} X \mid F_t]$, which represents the expected value of a discounted claim $B_t^{-1} X$ S at time T (conditional on the history up to time t), is also a martingale.

The MRT then leads us to the construction of the replicating portfolio, i.e. the appropriate v volumes φ_t of stock to hold. Since E and Z are both martingales, the MRT tells us that there is a previsible process φ_t such that $dE_t = \varphi_t\, dZ_t$ under measure **Q**.

This is important because if we are holding a volume φ_t of stock, then changes in the value of our stock and cash portfolio will match changes in the derivative's expected value, i.e. it is self-financing.

To complete the replication, the volume of cash needed is then $\psi_t = E_t - \varphi_t Z_t$.

The portfolio consisting of φ_t units of stock and ψ_t units of cash will always have value equal to the value of the derivative since its current value is equal to the value of the derivative and changes in the value of the components of the portfolio exactly match changes in the value of the derivative.

Hence we obtain the risk-neutral pricing formula for the value V_t of the derivative:

$$V_t = B_t E_Q[B_T^{-1} X | F_t] = \exp\left(-r(T-t)\right) E_Q[X|F_t].$$

[5]

[10 Marks]

__**Solution 7 :**__

i)

The main approach in every case is to express survival probabilities from the particular cohort of lives into a *survivor index* calculated up to some horizon T,

i.e. $S(t) = p(0, x).p(1, x)....p(t-1, x)$; for $t = 1, 2, ..., T$

where $p(k, x)$ is the one-period survival probability for group of lives x from time k to $k + 1$, calculated with reference to a specified relevant population.

The requisite derivative instruments are swaps, bonds and options (caps). Swaps and bonds hedge this risk outright, and hence there is no surplus gain to be generated if survival rates are lower than anticipated, whilst options allow the company to continue to take one-sided gain in such a case.

Swaps
A mortality swap of term T is an OTC contract that swaps a notional amount times the survivor index each period $t = 1, 2, \ldots, T$ for regular fixed or LIBOR-linked floating payments.
By receiving or paying mortality payments on this swap in the appropriate nominal amount, the company will receive payments that match its outflows on longevity risk.

Longevity Bonds
A longevity bond of maturity T is a tradable security that has coupons linked to the survivor index at times t = 1, 2, … , T.

By holding this bond in the appropriate nominal amount, the company will receive payments that match its outflows on longevity risk.

Principal-at-risk bonds
Instead of using the coupons to hedge the longevity risk, bonds can be constructed where the final principal at time t reflects the survivor index at time t.

Hedging is effected by issuing a strip of these bonds.

Survivor caps
Survivor caps are OTC contracts like interest-rate caps. A survivor caplet will pay at time t the maximum of $S(t) – K(t)$ and 0, where $K(t)$ is an agreed fixed strike amount.

A survivor cap of term T is simply a collection of survivor caplets with payment dates $t = 1, 2, \ldots, T$.

[5]

ii) In an OTC derivative contract, both parties undertake to pay each other cashflows at certain times. Cashflows due on the same date are usually offset, i.e. only the net balance passes between the parties in a single payment. Counterparty risk arises from the value of future cashflows being at risk to default, and the amount of exposure is often linked to (and hence varies with) underlying market variables such as interest rates, foreign exchange rates, equity prices etc.

If a particular firm has a counterparty that defaults, future payments will not be made, leading potentially to loss for the firm. However, there can only be a loss on a derivative contract if a counterparty defaults and the contract with them has positive value to the firm (i.e. net positive cashflows payable to the firm) at the time of default.

Some of the mitigants are:

- Break clauses – allowing a party to break the agreements say every 5 or 10 years in the event of a serious credit downgrade of the counterparty; also, could allow the contract to be re-negotiated at market rates periodically.

- Netting – ensure the derivative is subject to a legal agreement to net.
- Collateralisation – asking for temporary cash collateral to cover the mark-to-market of the trade, thereby ensuring that a positive mark-to-market value of the OTC trade can be recovered if the counterparty defaults since one side will always hold money to cover their potential loss; probably link thresholds to the credit rating of the counterparty.
- Diversification – using a range of banks to avoid concentration on any one.
- Design of contract – avoiding getting the market value too large.
- Rating quality – policy guidelines on the lowest allowable rating for a counterparty bank, for example S&P single A (or equivalent).
- Monitoring exposure – careful controls on the extent of counterparty risk.
- Credit insurance – using credit derivatives (e.g. credit default swaps), or seeking a parental or third party guarantee.

[4]

iii)
a) LPI is the same as RPI (Retail Price Indexation) inflation except that it is capped at a certain level (say 5%) and usually floored at zero.

Hence a company with an LPI-linked liability has basic RPI risk but has effectively bought a cap on RPI and sold a 0% floor.

To mitigate the risk, the company can find an LPI-linked hedge – take out an LPI swap, or buy an LPI bond.

Although inflation is capped under LPI, in a low inflation environment LPI and RPI are similar, hence an RPI hedge might be effective, as well as giving upside, but care must taken with an RPI hedge (or LPI hedge without the floor), as there is also a risk that inflation falls below zero, i.e. turns to deflation, because in that situation the 0% floor will kick in and the hedge will be mismatched.

Alternative hedges are: transfers of liabilities, securitisations, buyouts, reinsurance. There can be a second order cross-impact between LPI and longevity: inflating payments increases the exposure to longevity (by decreasing the effective discount rate), or if longevity is under-estimated then any LPI hedge will be insufficient.

b) Those with risk from inflation increasing are ones whose liabilities are inflation-linked; such as pension funds running Defined Benefit (DB) schemes, companies with fixed price contracts and costs that rise with inflation (e.g. energy companies), issuers of index-linked bonds or annuities.

Those with risk from inflation decreasing are ones whose assets are inflation-linked such as housing trusts, hospitals and schools with income linked to government contracts, companies with products or services that rise in line with inflation (e.g. shops, estate agents etc), pensioners in DB schemes.

Of these, those firms with inflation-linked income are very likely to sell inflation, and the pension funds are the main buyers.

[5]

[14 Marks]

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