Institute of Actuaries of India

Subject CT8 – Financial Economics

May 2015 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i. As per the first part of the B's statement, he believes that the Market demonstrates Efficiency in Strong form as the stock price reflects all the public and insider information
However, as per the latter part, he believes that technical analysis can lead to higher returns and this is possible only in inefficient markets.

If the markets demonstrate strong form of efficiency, the current market price also reflects all the historical prices too. Hence technical or fundamental analysis cannot yield higher returns as per this theory. Hence the two parts of the statements are mutually inconsistent. **[3]**

ii. Since the insider information yielded higher profits, the market does not show efficiency in strong form.
The markets can be efficient in weak form or semi - strong from or possibly even inefficient.
The exact form cannot be determined based on the data given in the question. **[2]**

iii. Effect of options: Status Quo Bias and Regret Aversion **[2]**

[7 Marks]

Solution 2:

i.

$$U'(w) = (-2)w^{-3} + 1$$

$$U''(w) = (6)w^{-4}$$

$$A(w) = \frac{-U''(w)}{U'(w)} = \frac{6w^{-4}}{2w^{-3}-1} = \frac{6}{2w-w^4}$$

$$A'(w) = -\frac{6}{(2w-w^4)^2} \times (2-4w^3)$$

A'(w) = 0 for $w^3 = 0.5$ or w = 0.793 → Constant ARA
A'(w) < 0 for w < 0.793 and → Decreasing ARA
A'(w) > 0 for w > 0.793 and → Increasing ARA

$$R(w) = wA(w) = \frac{6}{2-w^3}$$

$$R'(w) = \frac{18w^2}{(2-w^3)^2}$$

R'(w) is always > 0 for w>0. Hence the population exhibits increasing relative risk aversion. **[5]**

ii. Whether an investor will buy the govt. bond or stay invested in the company deposit will depend on the utility derived by the investor in either case.

Utility derived by the population by holding the Company deposits is:

$Ua(w) = 0.9\ ((1.2w)^{-2} + 1.2\ w) + 0.1(w^{-2} + w) = 0.725w^{-2} + 1.18w$

Utility derived by the population by buying the Central Bank's bonds is:

$Ub(w) = ((1.15w)^{-2} + 1.15\ w) = 0.756w^{-2} + 1.15w$

Ub>Ua if

$$0.756w^{-2} + 1.15w > 0.725w^{-2} + 1.18w$$

i.e.

$$0.03w^3 - 0.0311 < 0$$
$$w^3 - 1.037 < 0$$

the above equation has only one real root at (1.037)^(1/3) (the other two are complex)
i.e. above inequality is true for w < 1.01
Assuming that population is uniformly distributed across the wealth levels

there are 50% x 500 = 250 people with wealth (0.51, 1.01); hence a total of (100 + 250 = 350) people will subscribe the bond issue. **[4]**

iii. Since the utility is to be tested for the given level of wealth, w can be treated as constant while taking derivatives to determine the trend of U'.

<u>Equation 1</u>

$$\frac{\partial U(w,x)}{\partial x} = -\frac{e^{-\frac{x}{100}}}{100}$$

U'<0 for all x, Hence the Utility is reducing function of x. The utility derived for given level of wealth is lower for people of higher ages.
This function is exactly opposite to the government's intuition and does not capture the expected behaviour.

<u>Equation 2</u>

$$\frac{\partial U(w,x)}{\partial x} = \frac{1}{x+w}$$

U'>0 for all x, hence the Utility is increasing function of x. The utility derived for given level of wealth is higher for people of higher ages.
This function captures the expected behaviour. **[4]**

[13 Marks]

Solution 3:

i. Let S_t/S_0 follows lognormal distribution with parameters $\left(\mu - \frac{1}{2}\sigma^2\right)t$, and $\sigma^2 t$ such the expected return on a stock is μ and volatility is σ

This means Expected value of stock price at the end of first time step $= S_0 e^{\mu\delta t}$. On the tree the expected price at that time = $qS_0 u + (1-q)S_0 d$

In order to match the expected return on the stock with the tree's parameters we have
$qS_0 u + (1-q)S_0 d = S_0 e^{\mu\delta t}$.
Or $q = (e^{\mu\delta t} - d)/(u-d)$

Volatility σ of a stock price is defined so that $\sigma\sqrt{\delta t}$ is the standard deviation of the return on the stock price in a short period of time δt

The variance of stock price return is
$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2\delta t$

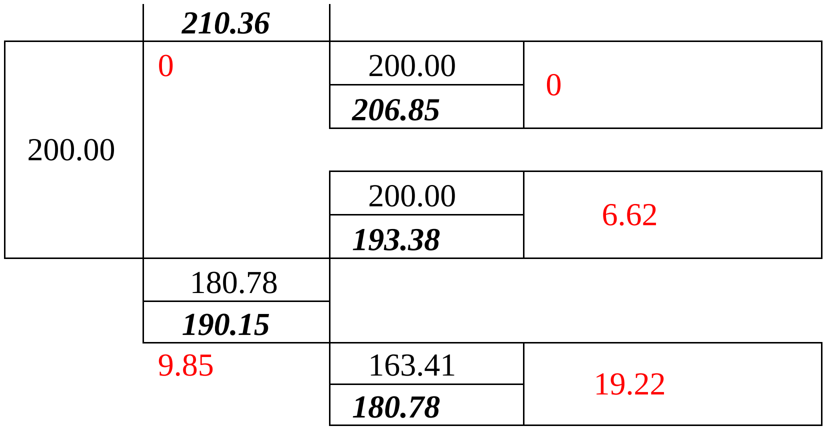
Substituting the value of q in the expression above we have
$e^{\mu\delta t}(u+d) - ud - e^{2\mu\delta t} = \sigma^2\delta t$

When higher powers of δt other than δt are ignored.
This implies u = $e^{\sigma\sqrt{\delta t}}$ and d = 1/u = $e^{-\sigma\sqrt{\delta t}}$ **[5]**

ii.

S	200
r	10%
σ	35%
T	2 months
t	1 month = 0.0833
u	1.1063
d	0.9039
q	0.5161
p=(1-q)	0.4839
K	200

t=0	t=1	t=2	Pay off @ K=200
		244.79	0
		221.26	
	221.26		



Note: Bold italics are geometric means of the stock price, reds are the payoffs.

The value of option at time 1 is

$(6.62 \times q + 19.22(1-q))e^{-0.1\times\frac{1}{12}} = 12.61$ this is more than the payoff at time 1. Hence the American option will not be exercised.

The value of put option at time 0 is

$12.61 \times e^{-0.1\times\frac{1}{12}} \times (1-q) = 6.05$ **[4]**

[9 Marks]

Solution 4:

K	320
S	300
r	6%
Volatility	$\sqrt{0.06}$ = 24.49%
T	1
Price of a call option c	$S_0N(d1) - Ke^{-rt}N(d2)$
d1	$\frac{\left(ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = 0.1039$
d2	$\frac{\left(ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T\right)}{\sigma T^{0.5}} = -0.1410$
N(d1)	0.5414
N(d2)	0.4439
Price of a call option	28.63
Price of a put option	30.00
Put call parity	$P + S_0 = Ke^{-rt} + C$

Similarly for variance 0.09 value = 36.54

Similarly for variance 0.12 value = 42.03

The expected value of the option
= 20% x 30+ 50% x 36.54 + 30% x 42.03 = 36.88 **[8 Marks]**

Solution 5:

i. The delta indicates that when the value of Euro exchange rate increases by $0.01, the value of the bank's position increases by 0.01*30,000=$300.

The gamma indicates that when the Euro exchange rate increases by $0.01, the delta of the portfolio decreases by 0.01*80,000=800. **[2]**

ii. For delta neutrality 30,000 euros should be shorted. **[1]**

iii. When the exchange rate moves up to 0.92, we expect the delta to move down by (0.92-0.88)*80,000 = 3,200. This means the new delta is 30,000-3,200 =26,800. **[1]**

iv. To maintain delta neutrality the bank has to unwind that much of euros in the portfolio. Hence a net of 26,800 euros have to be shorted. **[1]**

v. When a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is large movement in the underlying price. Hence the bank is likely to have lost money. **[1]**

[6 Marks]

Solution 6:

i. $dr(t) = \alpha(\mu - r(t))dt + \sigma(\sqrt{r(t)}d\widetilde{W}(t)$ **or**

$dr(t) = 0.2(0.08 - r(t))dt + 0.1(\sqrt{r(t)}d\widetilde{W}(t)$ **[0.5]**

ii. Revised SDE is

$dr(t) = \alpha(\mu - r(t))dt + \sigma(\sqrt{r(t)}d\widetilde{W}(t) + \varphi\, r(t)dt$

$dr(t) = (\alpha - \varphi)(\frac{\alpha\mu}{\alpha-\varphi} - r(t))dt + \sigma(\sqrt{r(t)}d\widetilde{W}(t)$

$dr(t) = (0.14)(0.1143 - r(t))dt + 0.1(\sqrt{r(t)}d\widetilde{W}(t)$ **[1.5]**

Hence the revised parameters are as follows:

α' = 0.14, μ' = 0.1143 σ' = 0.1

iii. Bond prices at time 5 and 10:

	Time 5	**Time 10**	**Marks**
$\theta = \sqrt{\alpha'^2 + 2\sigma'^2}$	0.199	0.199	1
$b(\tau) = \dfrac{2(e^{\theta\tau} - 1)}{(\theta + \alpha')(e^{\theta\tau} - 1) + 2\theta}$	3.494	4.975	2 (1 for each correct b(τ))

$a(\tau) = \frac{2\alpha'\mu'}{\sigma'^2}\ln\left(\frac{2\theta(e^{(\theta+\alpha')\tau/2})}{(\theta+\alpha')(e^{\theta\tau}-1)+2\theta}\right)$	-0.1581	-0.5058	2 (1 for each correct a(τ))
$\boldsymbol{B(t,T) = e^{a(\tau)-b(\tau)r(t)}}$	**0.6685**	**0.4257**	2 (1 for each correct B)

[7]

iv. Spot Rate: The spot rate R(t,T) is the constant force of interest applicable over the period from time t to time T that is implied by the market prices at time t.

$$R(t,T) = -\frac{1}{T-t}\ln\big(B(t,T)\big) = 8.54\%$$

Forward Rate: It represents the force of interest at which one can agree at time t to borrow or lend over the period from T to S (where t<T<S)

$$F(t,T,S) = \frac{1}{S-T}\ln\left(\frac{B(t,T)}{B(t,S)}\right) = \frac{1}{5}\ln\left(\frac{0.6685}{0.4357}\right) = 9.02\%$$

Instantaneous forward Rate: f(t,T) is the force of interest at future time T implied by the current market prices at time t.

$$f(t,T) = \lim_{S\to T} F(t,T,S) = -\frac{\partial \ln\,(B(t.T)}{\partial T}$$

[5]
[14 Marks]

Solution 7:

i. Expected cash-flow till t=1

$C(1) = 0.95(0.03 \times 100) + 0.05(S_0)$

Expected cash-flow till t=2

$$C(2) \quad = 0.95(0.03 \times 100) + 0.05(S_0) + 0.95^2(0.03 \times 100) + 0.95 \times 0.05(1.05 \times S_0)$$

Expected cash-flow till t=k

$$C(k) \quad = 0.95(0.03 \times 100) + 0.05(S_0) +$$

$0.95^2(0.03 \times 100) + 0.95 \times 0.05(1.05 \times S_0) +$

$\ldots +$

$0.95^k(0.03 \times 100) + 0.95^{k-1} \times 0.05(1.05^{k-1} \times S_0)$ **[4]**

ii. Fair value of the share will be same as the discounted value of the cash-flows at the end of each year.

Adding the discounting factor to the equation for C(k) gives the following:

$C(k) = 0.95(3)/1.08 + 0.05/1.08\ (S_0) +$

$0.95^2(3)/1.08^2 + 0.95 \times 0.05/1.08^2 \times (1.05 \times S_0) +$

$\ldots +$

$0.95^k(3)/1.08^k + 0.95^{k-1} \times 0.05/1.08^k(1.05^{k-1} \times S_0)$

$$C(\infty) = 3 \times \frac{0.95}{1.08} \times \sum_{i=0}^{\infty} \frac{0.95^i}{1.08^i} + \frac{0.95}{1.08} \times 0.05 \times S_0 \sum_{i=0}^{\infty} 0.95^i \times \frac{1.05^i}{1.08^i}$$

Applying the formula for summation of GP,

$$C(\infty) = 3 \times \frac{0.95}{1.08} \times \frac{1}{1 - \frac{0.95}{1.08}} + \frac{0.95}{1.08} \times 0.05 \times S_0 \frac{1}{1 - \frac{0.95 \times 1.05}{1.08}} \quad \ldots (1)$$

$$C(\infty) = 21.923 + 0.5758\, S_0$$

As the pricing of share is fair, $S_0 = C(\infty)$

$S_0 = 21.923 + 0.5758\, S_0$

$i.e. S_0 = Rs. 51.66$ **[5]**

[9 Marks]

Solution 8:

i. Mean and s.d. of the investor's portfolio (P):

$\mu_P = a\mu_B + (1-a)\mu_M$

$\sigma_P^2 = a^2\sigma_B^2 + (1-a)^2\sigma_M^2 + 2a(1-a)\sigma_{BM}$

σ_{BM} is the covariance of B to M. **[2]**

ii. The slope of the frontier on μ-σ space traced by the investor's portfolio at a=0 is:

$\frac{d\mu}{d\sigma} = \frac{(d\mu/da)}{(d\sigma/da)}$ …(Eq1)

Differentiating μ w,r,t. a $\frac{d\mu}{da} = \mu_B - \mu_M$

Differentiating σ^2 w,r,t. a

$2\sigma_P \frac{d\sigma}{da} = 2a(\sigma_B^2) - 2(1-a)(\sigma_M^2) + 2(1-2a)\sigma_{BM}$

At a=0, P is same as M

hence above equation becomes:

$2\sigma_M \frac{d\sigma}{da} = -2(\sigma_M^2) + 2\sigma_{BM}$

$\frac{d\sigma}{da} = \frac{\sigma_{BM} - (\sigma_M^2)}{\sigma_M}$

Substituting in Eq 1

$\frac{d\mu}{d\sigma} = \frac{(\mu_B - \mu_M)}{\left(\frac{\sigma_{BM} - (\sigma_M^2)}{\sigma_M}\right)}$ **[4]**

iii. Equating the slope in above section to the slope given in question

$$\frac{(\mu_B - \mu_M)}{\left(\frac{\sigma_{BM} - (\sigma_M^2)}{\sigma_M}\right)} = \frac{\mu_M - \mu_Z}{\sigma_M}$$

$$(\mu_B - \mu_M) = (\mu_M - \mu_Z)\frac{\sigma_{BM}}{\sigma_M^2} - (\mu_M - \mu_Z)$$

Simplified equation:

$\mu_B = \mu_Z + (\mu_M - \mu_Z)\frac{\sigma_{BM}}{\sigma_M^2}$ **[2]**

iv. The equation derived in part (iv) is Capital Market Line for a ‘Zero Beta Model’.

Uses

The Capital Asset Pricing Model assumes existence of a risk free security whereas in real life, no security is truly risk free.

A portfolio ‘z’ that has no co-relation with the market portfolio i.e. β_z is zero. Hence this is used to estimate returns that are independent of the market risk.

[2]

[10 marks]

Solution 9:

i. S_t follows Geometric Brownian motion with drift μ and volatility σ

This implies

$$S_t = S_0 e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W t}$$

Where $W_t \sim N(0, \sigma^2 t)$

Implies $\frac{St}{S_0} = e^{\mu t - \frac{1}{2}\sigma^2 t + \sigma W t}$

Or $ln\frac{St}{S_0} = \mu t - \frac{1}{2}\sigma^2 t + \sigma W_t$

Or $ln\frac{St}{S_0} \sim N(\mu t - \frac{1}{2}\sigma^2 t, \sigma^2 t)$

Or $\frac{St}{S_0}$ follows lognormal distribution with parameters $(\mu t - \frac{1}{2}\sigma^2 t, \sigma^2 t)$

If $X \sim N(0, \sigma^2)$, then $Z = e^X \sim ln(\mu, \sigma^2)$

$E(Z) = e^{\mu + \frac{1}{2}\sigma^2}$ and $Var(Z) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$

Hence $E\left(\frac{St}{S_0}\right) = e^{\mu t - \frac{1}{2}\sigma^2 t + \frac{1}{2}\sigma^2 t} = e^{\mu t}$

This means $E(St) = S_{0e^{\mu t}}$

$Var\left(\frac{St}{S_0}\right) = (e^{\sigma^2 t} - 1)e^{2(\mu t - \left(\frac{1}{2}\right)\sigma^2 t) + \sigma^2 t} = (e^{\sigma^2 t} - 1)e^{2\mu t}$

$Var(St) = S_0^2(e^{\sigma^2 t} - 1)\, e^{2\mu t}$

Confidence limit for S_T:

$\ln(S_t) \sim N\left(\ln(S_0) + \mu T - \frac{1}{2}\sigma^2 T, \sigma^2 T\right)$

95% confidence interval for $Ln(S_T)$ is therefore

$Ln(S_0) + \left(\mu - \left(\frac{1}{2}\right)\sigma^2\right)T - 1.96\sigma\sqrt{T}$ and

$Ln(S_0) + \left(\mu - \left(\frac{1}{2}\right)\sigma^2\right)T + 1.96\sigma\sqrt{T}$

This means 95% confidence interval for S_T is

$$\left(e^{\ln(S_0) + \left(\mu - \left(\frac{1}{2}\right)\sigma^2\right)T - 1.96\sigma\sqrt{T}}, e^{Ln(S_0) + \left(\mu - \left(\frac{1}{2}\right)\sigma^2\right)T + 1.96\sigma\sqrt{T}}\right)$$

[7]

ii. Price of a security under risk neutral measure is given by $V_t = e^{-r(T-t)} E_Q[X| F_T]$ where r is the risk free rate of interest

Where $X = S_T - K$

Substituting this in the equation above we get,

$V_t = e^{-r(T-t)} E_Q[S_T - K| F_T] = e^{-r(T-t)}(E_Q[S_T| F_T] - K)$

As S follows Geometric Brownian motion,

$E_Q[S_T| F_T] = e^{\ln(S_t) + \left(r - \left(\frac{1}{2}\right)\sigma^2\right)(T-t) + \left(\frac{1}{2}\right)\sigma^2 (T-t)} = S_t e^{r(T-t)}$

$V_t = e^{-r(T-t)}(E_Q[S_T| F_T] - K) = S_t - Ke^{-r(T-t)}$ **[4]**

iii. From Black Scholes PDE, we have

$$\theta + rS\Delta + \left(\frac{1}{2}\right)\sigma^2 S^2 \Gamma = rV$$

Where $\theta = \frac{\partial V}{\partial t} = -rKe^{r(T-t)}$

$$\Delta = \frac{\partial V}{\partial S} = 1$$

And $\Gamma = \frac{\partial \Delta}{\partial S} = 0$

Hence, substituting the above values/expression we get

$LHS = -rKe^{r(T-t)} + rS = r\left(S - Ke^{r(T-t)}\right)$ this satisfies the Black Scholes PDE only if $V = S - Ke^{r(T-t)}$ **[3]**

[14 Marks]

<u>Solution 10:</u>

i.

$$f(t, S_t) = \left(S_t^{-\frac{2r}{\sigma^2}}\right)$$

Assumption: stock price follows Geometric Brownian motion given by

$dS_t = \mu S_t \, dt + \sigma S_t dW_t$ with drift parameter μ and volatility σ

where dW_t is a Weiner process

$$\frac{\partial f}{\partial S} = \left(-\frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}-1}$$

$$\frac{\partial^2 f}{\partial^2 S} = \left(-\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}-1\right) S^{-\frac{2r}{\sigma^2}-2}$$

$$= \left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right) S^{-\frac{2r}{\sigma^2}-2}$$

Using Ito calculus, $df = \frac{\partial f}{\partial S} dS + \frac{1}{2}\frac{\partial^2 f}{\partial^2 S}(dS)^2$

Therefore $df = \left(-\frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}-1}(\mu S dt + \sigma S dW) + \frac{1}{2}\left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right) S^{-\frac{2r}{\sigma^2}-2}(\sigma^2 S^2 dt)$
$= \left(-\frac{2r}{\sigma^2}\mu + \frac{2r}{\sigma^2} r + r\right) f.dt - \frac{2r}{\sigma} f.dW$
…(1.5)
The process is a martingale if drift term is 0.

This means $-\frac{2r}{\sigma^2}\mu + \frac{2r}{\sigma^2} r + r = 0$
Or $\mu = \left(\frac{2r}{\sigma^2}+1\right)\frac{\sigma^2}{2}$

For the process to be price of a traded security, this should satisfy the Black Scholes PDE.

We have
$\frac{\partial f}{\partial S} = \left(-\frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}-1}, \frac{\partial^2 f}{\partial^2 S} = \left(-\frac{2r}{\sigma^2}\right)\left(-\frac{2r}{\sigma^2}-1\right) S^{-\frac{2r}{\sigma^2}-2}, \frac{\partial f}{\partial t} = 0$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} rS + \left(\frac{1}{2}\right)\frac{\partial^2 f}{\partial^2 S}\sigma^2 S^2 = rS_t^{-2r/\sigma^2} = rf$$

Hence Black Scholes equation is satisfied, so the expression can be price of a traded security.

[6]

ii. We want to solve the equation:
$dX_t = -\gamma X_t dt + \sigma dB_t$ or $dX_t + \gamma X_t dt = \sigma dB_t$
Multiplying through by $e^{\gamma t}$ – the integrating factor and then changing the dummy variable to s gives:
$e^{\gamma s} dX_s + \gamma e^{\gamma s} X_s ds = \sigma e^{\gamma s} dB_s$

The left-hand side is now the differential of a product. So we have:
$d(e^{\gamma s} X_s) = \sigma e^{\gamma s} dB_s$
Integrating from 0 to t we get:

$(e^{\gamma t}X_t - e^{\gamma 0}X_0) = \sigma \int_0^t e^{\gamma s} dBs$

Rearranging,
$X_t = X_0 e^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-s)} dBs$

$E(X_t) = X_0 e^{-\gamma t}$ which implies that $E(X_t)$ is not equal to X_0 for $\gamma > 0$ and hence not a martingale. **[4]**

[10 Marks]

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