Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2015 Examination

INDICATIVE SOLUTION

Solution 1:

(i)

$$e^{-\frac{\delta}{4}} = 1 - \frac{0.08}{4}$$

$\delta = 0.080811$ **[1]**

(ii)

$$(1+i)^{-1} = \left(1 - \frac{0.08}{4}\right)^4 = 0.92237$$

$i = 0.084166$ **[1]**

(iii)

$(1 - \frac{d^{(12)}}{12})^{12} = (1 - \frac{0.08}{4})^4$ = 0.92337
$d^{(12)} = 0.080539$ **[1]**

[3 Marks]

Solution 2:

(i) Call option: A call option gives you the right, but not the obligation, to buy a specified asset on a set date in future for a specified price

Put option: A put option gives you the right, but not the obligation, to sell a specified asset on set date in future for a specified price.

[2]

(ii) Debentures are a part of the loan capital of the company. The term "loan capital" usually refers to long term borrowings rather than short term. The issuing company provides some form of security to the holders of the debenture. This is usually in the form of a floating charge against the assets of the company.

Unsecured loan stocks have no explicit assets backing them and holders rank alongside other unsecured creditors. The Yields will be higher than on the debentures to reflect the higher risk of default.

[2]

[4 Marks]

Solution 3:

(i)

5.91% > 7% x (1 - 0.25)
Assume stock is redeemed as late as possible (i.e. 1 April 2021)

For INR 100 nominal:
P '= Price at 1 April 2002

$= 0.75 \text{ x } 7\, a_{\overline{19|}}^{(2)} + 100 \text{ x } V^{19}$ at 6%

= 0.75 x 7 x 1.014782 x 11.1581 + 100 x 0.33051
= 59.446 + 33.051 = 92.497

So price at 1 July 2002, P is
$P = 92.497 \text{ x } (1.06)^{3/12} = 93.854$
and price of INR10,000 nominal is INR 9,385.40 **[2]**

(ii)
4.94% < 7% x (1 -0.25)
Assume stock is redeemed at earliest possible date (i.e. 1 April 2015)
For 100 nominal:
$P = 0.75 \text{ x } 7\ a_{\overline{5|}}^{(2)} + 100v^5$ at 5%

= 5.25 x 1.012348 x 4.3295 + 100 x 0.78353
= 101.364
Hence price of INR 10,000 nominal is INR 10,136.40 **[2]**

[4 Marks]

Solution 4:

Total accumulated value of,

$$\rho(t) = \begin{cases} 10 \mp 10t & 3 < t < 4 \\ 50 - t^2 & 5 < t < 6 \end{cases}$$

at time, t = 10

$$= \int_3^4 (10 + 10t) \exp\left(\int_t^4 (0.01 + 0.01s) ds\right) \exp\left(\int_4^6 (0.15s - 0.003s^2)\, ds + \int_6^{10} 0.06\, ds\right) dt + \int_5^6 (50 - t^2) \exp\left(\int_t^6 0.15 - 0.003s^2\right) \text{ds} \exp\left(\int_6^{10} 0.06\, ds\right) dt$$

$$= -1000 \int_3^4 (-0.01 - 0.01t) \exp(0.04 + 0.08 - 0.01t - 0.005t^2) \exp(0.388) dt - \frac{1000}{3} \int_5^6 (-0.15 + 0.003t^2) \exp(0.9 - 0.216 - 0.15t + 0.001t^3) \exp(0.24) dt$$

$$= -1000\, (e^0 - e^{0.045}) e^{0.388} - \frac{1000}{3}\, (e^0 - e^{0.059}) e^{0.24}$$

= 67.846 + 25.753 = 93.60

[6 Marks]

Solution 5:

(i)
Let i %= money rate of return
Then
9900 = $800a_{25}^{(2)}$ +11,000v^{25} - 200$v^{1/4}$ a_{25} -0.3 (11,000- 9,900)$v^{(25+\frac{1}{4})}$

1st approximation:
$i = \frac{600+\frac{11000-300-990}{25}}{9900}$
= 6.4%

At 7%: RHS = 800 ×1.017204 ×11.6536 + 11000 ×0.18425 -200$v^{1/4}$ × 11.6536 -330$v^{(25+)}$
= 9158.61
Try 6%: RHS = 800 × 1.014782 × 12.7834 + 11000 ×0.23300 - 200$v^{1/4}$ × 12.7834 -330 $v^{(25+1/4)}$
= 10345.41 **[6]**
$i = 0.07 - \frac{9900-9158.61}{10345.41-9158.61}$ ×0.01
i= 0.0638 i.e. 6.38% p.a.

Hence, if r%= real return
$1 + r = \frac{1.0638}{1.03}$ = 1.0328
real return = 3.28% p.a. **[2]**

(ii)
If tax were collected 2 months later (i.e. 1 June rather than 1 April) then investor is deferring paying the tax. Hence, real return would be higher than 3.28% p.a. **[1]**

[9 Marks]

Solution 6:

(i) $i1$ = 6%
$p2$ = 6(1.06)$^{□□}$ + 106(1 + i2)$^{□□}$
$p2$ = 6a_2 □□100v^2@ 6.3%
(1 + i2)$^{□□}$ = 6a_2 □□100$v^2$106 -6(1.06)$^{□□□}$/106
i2 = 6.30907%

$p3$ = 6(1.06)$^{□□□}$+ 6(1.0630907)$^{□□□}$ + 106(1 + i3)$^{□□□}$
$p3$ = 6a_3□□100v^3
i3 = 6.62476%

(P(t) is the price of the bond with term t,)
[4]

(ii)$f_{0,1}$ = 6%
(1 + $f_{1,1}$) (1.06) = $(1.0630907)^2$
$f_{1,1}$ = 6.61904%
(1 + $f_{2,1}$) $(1.0630907)^2$ = $(1.0662476)^3$
$f_{2,1}$ = 7.25896%
[4]

(iii) The accumulation factors related to spot rates are geometric averages of accumulation factors related to the forward rate for the same year as well as for all those relating to all previous years. Therefore, if forward rates are increasing, spot rates, being an average of a spot rate for the given year and for previous years, will increase more slowly.

[1]

[9 Marks]

Solution 7:

(i) Let j denote the mean yield, then
$1 + j = \exp\left[\mu + \sigma 2/2\right] = 1.0757305$
$j = 0.0757305$

We require
$20{,}000\ E(X10) + 150{,}000\ E(S10)$
$= 20{,}000 s_{10}$ □□150,000 (1 □□j)^{10} at rate j%
$= 20{,}000 \times 14.1961 + 311{,}261.98$
$= 595{,}183.99$

where $X10$ represents the accumulation after 10 years of INR1 p.a. paid in arrears for 10 years and $S10$ represents the accumulation after 10 years of INR1 paid now. **[5]**

(ii) We require Pr(Z . $S10 \geq$ □600,000) = 0.99
where Z = single amount paid now
Pr [$S10 \geq$ □600,000 /z] = 0.99

Now { log $S10$ – 10 μ / σ √10} ˜ N(0,1)
So we want
Φ {log (600,000/z) -10 μ/σ √10} = 0.01

So, from tables,
log (600,000/z) -10 μ/σ √10= - 2.326
So 600,000 /Z = exp□□2.326 σ √10□□10 μ□□= 1.139112
Z = 526,726.25 **[6]**

[11 Marks]

Solution 8:

(i) APR is the interest rate that solves equation of value. Hence, APR, I, is
$20{,}000 = 12 \times 427.90\ a_5^{(12)}$ => $a_5^{(12)} = 3.89499$

APR is usually double the flat rate of interest. Hence,
Flat Rate = $\frac{total\ interest}{loan \times total\ years}$ = $\frac{5\times12\times427.90-20000}{20000\times5}$ = 5.67%

Thus, APR is likely to be around 11%.

At i=11%, $a_5^{(12)} = 3.87872$

At i= 10%, $a_5^{(12)} = 3.96154$

Interpolating, APR, I is

$$\frac{i - 10\%}{3.89499 - 3.96154} = \frac{11\% - 10\%}{3.87872 - 3.96154}$$

i = 10.8%

At i = 10.8%, $12 \times 427.90\ a_{5}^{(12)} = 20{,}000.2$

At i = 10.9%, $12 \times 427.90\ a_{5}^{(12)} = 19{,}958.3$

i = 10.8% is more close. Hence, APR is 10.8% **[5]**

(ii) APR = 10.8%

After 1 year, capital left to pay = $12 \times 427.90\ a_{4@10.8\%}^{(12)} = 16{,}775.98$

Let t denote the term of the new loan, thus

$$16{,}775.98 = 12 \times 274.49\ a_{t}^{(12)}$$

$$16{,}775.98 = 3293.88\ \frac{1 - 1.108^{-t}}{12\times(1.108^{\frac{1}{12}} - 1)}$$

$1.108^{-t} = 0.4754$

t = 7.25 years

The original loan had 4 years of payments remaining, thus

Total Interest = $4 \times 12 \times 427.90 - 16775.98 = 3763.22$

New loan has term of 7.25 years, hence

Total Interest = $7.25 \times 12 \times 274.49 - 16775.98 = 7104.65$

Thus, Mr. & Mrs. Jones will pay 3341.43 (= 7104.65 – 3763.22) more interest under the restructured loan.

[6]
[11 Marks]

Solution 9:

(i) Work in 000's

PV of liabilities = $95 \times a_{20} + 5(Ia)_{20}$ at 7%
$= 95 \times 10.5940 + 5 \times \{(\ddot{a}_{20} - 20 v^{20})/i\}$
= 1006.43 + 5 x (11.3356 - 20 x 0.25842)/0.07
= 1446.94

Numerator of duration

$= 95(Ia)_{20} + 5(1.v + 2.2v^2 + \ldots\ldots + 20.20v^{20})$
$= 95 \times 88.1029 + 5/(1-v) \times \{2(Ia)_{20} - a_{20} - 400 v^{21}\}$
$= 8369.78 + 5 / 0.06542 \times \{ 2 \times 88.1029 - 10.5940 - 400 \times 0.24151\}$
$= 8369.78 + 5274.21 = 13,643.99$

Hence, duration of liabilities = 13643.99 /1446.94 = 9.43 years

[6]

(ii) Let nominal amount of 25-year stock be A and nominal amount of 12-year stock be B.

Then PV of assets = $A v^{25} + B \times 0.08 \times a_{12} + B v^{12}$
(= PV of liabilities = 1446.94) ----------------------(1)

and duration of assets =
$25A v^{25} + B \times 0.08 \times (Ia)12 + 12 B v^{12} / 1446.94$
= duration of liabilities = 13643.99 /1446.94 -------------- (2)
From (1),
1446.94 = A x 0.18425 + B x 0.08 x7.9427 + B x 0.44401
1446.94 = 0.18425A + 1.07943B ----------------------------(3)
From (2),
13643.99 = 25 x0.18425 x A + { 0.08 x [(1.07 x7.9427 -12 x 0.44401)/0.07] + 12 x 0.44401 } x B
13643.99 = 4.60625A + (3.62351 + 5.32812) B
= 4.60625A + 8.95163B -------------------------(4)
Now,
4.60625/0.18425 x (3)

36173.50 = 4.60625A + 26.98575B
22529.51 = 18.03412B
B = 1249.27
A = 534.27

Hence, amount invested in each security is:

Stock A: $534.27 \times v^{25} = 98.44 = 98,440$
Stock B: $1249.27 \times 0.08 \times a12 + 1249.27v^{12}$
= 1249.27 x1.07943 = 1348.50 = 1,348,500

[9]
[15 Marks]

Solution 10:

(i)
a. The discounted payback period is the smallest time t for which the present (or accumulated) value of the returns up to time t exceeds the present (or accumulated) value of the costs up to time t.

b. The payback period is the same as the discounted payback period, except that the present value calculation (or accumulation) is carried out using an interest rate of 0. In other words, it is the earliest time for which the monetary value of the returns exceeds the monetary value of the costs.

[2]

(ii) Unlike the NPV, neither the DPP nor the PP give any indication of how profitable a project is, as they ignore cash flows after the accumulated value of zero is reached.

There may not be one unique time when the balance in the investor's account changes from negative to positive. However, the NPV can always be calculated.

The PP can give misleading results, as it does not take into account the time value of money.

Hence, NPV is a superior measure as compared to DPP or PP.

[3]

(iii)

a. Internal rate of return

The IRR for Project A is the rate of interest that satisfies the equation of value:

$- 170{,}000 - 20{,}000v - 10{,}000v^2 + 20{,}000v + 20{,}000v^2 + 200{,}000v^3 = 0$

Simplifying this and expressing it in £000s:

$10v^2 + 200v^3 = 170$

By trial and error, we find that:

i = 7%; $10v^2 + 200v^3 = 171.99$

i = 8%; $10v^2 + 200v^3 = 167.34$

Using interpolation:

$$i = 7 + \frac{170-171.99}{167.34-171.99} * (8-7) = 7.4\%$$

For Project B, the income (paid at the end of each of the first 6 years) represents 7% of the initial capital, which is returned at the end of the 6 years.

So we can see immediately that the IRR for Project B is exactly 7%.

Alternatively:

$14a_6 + 200v^6 = 200$ **[5]**

b. Net Present Value

The net present values are found by discounting the payments at 6%.

Project A: NPV = $- 170{,}000 + 10{,}000v^2 + 200{,}000v^3$ = £6,823.82

Project B: NPV = $- 200{,}000 + 14{,}000a6| + 200{,}000v^6$ = £9,834.65 **[2]**

c. Maximum Interest Rate beyond which the projects are unprofitable

Each project will be profitable if the rate of interest at which funds can be borrowed is less than the internal rate of return. So, to be profitable, Project A requires a borrowing rate less than 7.4% and Project B requires a rate lower than 7%.

Other factors to be considered include:

- Project A does not provide any net income during the first year.
- The lower internal rate of return for Project B applies for a longer period.
- The rates of interest available in years 4 to 6 (ie after Project A has finished) will affect the comparison between the accumulated profits at the end of year 6 (ie when Project B finishes).
- The risk associated with the receipt of income. If Project A involves greater uncertainty or risk, it may not be accepted even though it has a higher IRR.

[2]
[14 Marks]

Solution 11:

(i)

a. Time weighted rate of return:

For Manager A:

$$(1+i)^3 = \frac{88}{80} \times \frac{123.6}{103} \times \frac{172.32}{143.6} = 1.584 \quad => i = 16.6\%$$

For Manager B:

$$(1+i)^3 = \frac{168}{140} \times \frac{219.6}{183} \times \frac{263.56}{239.6} = 1.584 \quad => i = 16.6\%$$

b. Money weighted rate of return:

For Manager A:

$$80(1+i)^3 + 15(1+i)^2 + 20\,(1+i) = 172.32$$

Using binomial approximation to obtain first guess,

80(1+3i) + 15(1+2i) + 20(1+i) = 172.32

i = 19.77%

Now,

i = 19% => LHS = 179.85
i = 18% => LHS = 175.93
i = 17% => LHS = 172.06
Using linear interpolation,

$$i = 17\% + \frac{172.32-172.06}{175.93-172.06} \times (18\% - 17\%) = 17.1\%$$

Similarly for manager B:

$$140(1+i)^3 + 15(1+i)^2 + 20\,(1+i) = 263.56$$

Using binomial approximation to obtain first guess,

140(1+3i) + 15(1+2i) + 20(1+i) = 263.56

i = 18.84%

Now,

i = 18% => LHS = 274.51
i = 17% => LHS = 268.16
i = 16% => LHS = 261.91

Using linear interpolation,

$$i = 16\% + \frac{263.56-261.91}{268.16-261.91} \times (17\% - 16\%) = 16.3\%$$ **[8]**

(ii) For manager A, the growth in cash flows each year is 10%, 20% and 20%.

For manager B, the growth in cash flows each year is 20%, 20% and 10%.

Both managers have overall same growth rates. Hence TWRR is same for both the managers. However, manager A performed worst in first year, whereas manager B performed worst in the last year.

MWRR puts more emphasis on the money. Since for both the managers, the greatest money in the fund occurs in the third year, the MWRR gives the 3rd year more weight than others. Hence, manager B's poor performance in the 3rd year will be more penalised. Due to this, manager B has lower MWRR.

[3]

(iii) MWRR is higher for manager A than manager B, whereas TWRR is same for both the managers. The TWRR is considered the better measure of performance as it ignores the effects of the cash flows, which are beyond managers' control.

On this basis, both the fund managers should be considered as having performed equally.

[2]
[13 marks]

****