Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2014 Examinations

INDICATIVE SOLUTIONS

Solution 1 :

The rate of interest for the first 7 yrs. is 8% per annum convertible monthly. The Half-yearly effective interest is: $= (1+\frac{0.08}{12})^6 - 1$

= 4.067%

The effective half-yearly interest for next 8 years is 3%.

The accumulated value at the end of 15 years is:

$$= 2000\ddot{s}_{\neg 14}@4.067\% \times (1.03)^{16}@3\% + 2000\ddot{s}_{\neg 16}@3\%$$

$$\ddot{s}_{\neg 14}@4.067\% = 19.1236$$

$$\ddot{s}_{\neg 16}@3\% = 20.7615$$

$$= 2000 \times 19.1236 \times 1.6047 + 2000 \times 20.7615$$

= ₹102,898.28

[4 Marks]

Solution 2 : The accumulated value of cash flow stream is:

$$\int_0^8 (4+0.5t)\{e^{\int_t^8 (0.04+0.005s)ds}\}dt$$

$$= \int_0^8 (4+0.5t)\exp\left[0.04s + \frac{0.005s^2}{2}\right]_t^8 dt$$

$$= \int_0^8 (4+0.5t)\exp\left[0.32 + 0.16 - \left(0.04t + \frac{0.005t^2}{2}\right)\right]dt$$

$$= \int_0^8 (4+0.5t)\exp\left[0.48 - \left(0.04t + \frac{0.005t^2}{2}\right)\right]dt$$

Let $f(x) = 0.48 - (0.04t + \frac{0.005t^2}{2})$

The above integral can be written as

$$= \int_0^8 -100 f'(x)\exp[f(x)]\,dt$$

$$= -100exp\left[0.48 - \left(0.04t + \frac{0.005t^2}{2}\right)\right]_0^8$$

$$= -100(e^0 - e^{0.48})$$

=61.60

[5 Marks]

Solution 3 :

a. An equation of value expresses the equality of the present value of positive and negative (or incoming and outgoing) cash flows that are connected with an investment project, investment transaction etc.

[1]

b.

$$4000 = 400 \times a_{\neg 10} + x\% \times 4000 \times v^{10} @ 7\%$$

$$x = \frac{4000 - 400 \times 7.0236}{4000 \times 0.50835}$$

$$x = 58.55\%$$

[3]

[Total Marks-4]

Solution 4 :

i. The first investor buys the government bill at 95 and sells at 98.42 after 60days. Working in years, if the annual rate of interest earned is i_1, then

$$95(1 + i_1)^{\frac{60}{365}} = 98.42 \quad =\gg \quad i_1 = 24\%$$

The second investor buys the bill at 98.42 and redeems at 100 after 30 days. If the annual rate of interest earned is i_2 then

$$98.42(1 + i_2)^{\frac{30}{365}} = 100 \quad =\gg \quad i_2 = 21.38\%$$

The first investor earned a higher rate of interest than the second investor.

[3]

ii. The discount factor for 91 days under simple discount is:

$$(1 - nd) \quad = \left(1 - \frac{91}{365} \times 0.09\right) = 0.9775$$

Equating this to effective discount factor:

$$v^{\frac{91}{365}} = 0.9775 \quad =\gg (1 + i)^{\frac{-91}{365}} = 0.9775 \quad =\gg i = 9.53\%$$

(Student can use 1/4th of a year in the above calculations as approximation)

[3]

[Total Marks-6]

Solution 5 :

i. Let i_t be the gross redemption yield (spot yield) over *t* years:

For one year: yield is 10%, thus $i_1 = 0.1$

For two years: $(1+i_2)^2 = 1.1 \times 1.09$, thus $i_2 = 0.094988$

For three years: $(1+i_3)^3 = 1.1 \times 1.09 \times 1.08$, thus $i_3 = 0.089969$

For four years: $(1+i_4)^4 = 1.1 \times 1.09 \times 1.08 \times 1.07 = 0.084942$

[4]

ii. The Price of the bond is $7\,\{(1.1)^{-1} + (1.094988)^{-2} + (1.089969)^{-3}\} + 107 \times (1.084942)^{-4}$

$= 17.60758 + 77.22495 = 94.83253$

To find the gross redemption yield on the bond, we have

$94.83253 = 7\, a_{\overline{4}|} + 100\, v^4$

At 9%, from Tables, $a_{\overline{4}|} = 3.2397$ and $v^4 = 0.70843$ and at 9% , the R.H.S of the above equation is 93.5209

At 8%, from Tables, $a_{\overline{4}|} = 3.3121$ and $v^4 = 0.73503$ and at 8%, the R.H.S of the above equation is 96.6877

By interpolation,

$$i = 0.09 - 0.01 \times \frac{(94.83253 - 93.5209)}{(96.6877 - 93.5209)}$$

$= 0.09 - 0.004142 = 0.085858 \quad = 8.58\%$

[3]

[Total Marks-7]

Solution 6 :

i. The running yield on the property is defined as the rental income net of all management expenses divided by the cost of buying the property gross of all purchase expenses.

[1]

ii. The running yield from the property investments will normally be higher than that for ordinary shares for the below mentioned reasons

:

1. Dividends usually increase annually, whereas rents are reviewed less often;
2. Property is much less marketable;

3. Expenses associated with property investment are much higher;
4. Large, indivisible units of property are much less flexible; and
5. On an average, dividends will tend to increase more rapidly than rents, as dividends benefit from returns arising from the retention of profits and their reinvestment within the company.

[2]
[Total Marks-3]

Solution 7 : The minimum effective yield required is

For i = 0.08, from tables $i^{(12)} = 0.077208$

By employing the capital gains test,

$i^{(p)} > (1-t_1)\dfrac{D}{R}$ Where t_1 is the income tax rate

$(1 - 0.1)\dfrac{0.09}{1.09} = 0.074311$

As $i^{(12)} > 0.074311$, there is Capital gain on the investment

Also, we should assume that the loan issued is redeemed as late as possible i.e. 25 years from the date of issue to obtain minimum yield.

Consider the price of the loan as P:

$P = 5,00{,}000 \times 0.09 \times 0.9 \times a_{\overline{25}|}^{(12)}$
$+ (5,45{,}000 - 0.2(5,45{,}000 - P)) \times v^{25}$ at 8%

Considering $a_{\overline{25}|}^{(12)} = i / i^{(12)}\, a_{\overline{25}|}$

$= 40500 \times 1.036157 \times 10.6748 + (436000 + 0.2P) \times 0.14602$

P = ₹5, 27,016.85

[8 Marks]

Solution 8 :

i. A *futures contract* is a standardized, exchange tradable contract between two parties to trade a specified asset on a set date in the future at a specified price.

A *forward contract* is an agreement made between two parties under which one party agrees to buy from the other party a specified amount of an asset at a specified price, called the *forward price*, on a specified future date.

[2]

ii. The current value of the forward price of the old contract is:

$$75(1.08)^{8} – 15(1.08)^{-4} - 20(1.08)^{-8} \quad \text{--- (1)}$$

Whereas the current value of the forward price of the new contract is:

$$175 - 15(1.08)^{-4} - 20(1.08)^{-8} \quad \text{--- (2)}$$

Hence, the current value of old forward contract is:

(2) – (1) = ₹36.18

Alternatively, the value can be directly arrived as:

$$175 - 75(1.08)^{8} = ₹36.18 \quad (3)$$

(The dividend is not relevant in this calculation.)

[3]

iii. Assuming 'no arbitrage':

The Present value of the payment is:

$100\ v^{0.25}$ (at 8%) + $100v^{0.75}$ (at 9%)

= 100 x (0.98094+0.93741) = 191.83

Hence the forward price is: F = (1200-191.83) x 1.09 = ₹1098.90 **[4]**

[Total Marks-9]

<u>Solution 9</u> :

i. Let the accumulation of an investment of 1 at the beginning after 3 years be S_3.

Let the annual effective interest rate for the year t be i_t.

Expected annual interest rate in each of the three years is:

Year 1, $E(i_1) = 1/3\ (0.04 + 0.05 + 0.06) = 0.05$

Year 2, $E(i_2) = 0.4 \times 0.08 + 0.6 \times 0.05 = 0.062$

Year 3, $E(i_3) = 0.9 \times 0.07 + 0.1 \times 0.05 = 0.068$

As the annual effective interest rate in any year is independent of the interest rate in any other years,

$E(125000\ S_3) = 125000\ E(S_3) = 125000\ E(1+i_1)\ E(1+i_2)\ E(1+i_3)$

= 125000 x 1.05 x 1.062 x 1.068 = ₹1, 48,865.85

[5]

ii. Assuming the interest rates declared each year is the maximum probable rates, then the accumulated amount would be

125000 x 1.06 x 1.08 x 1.07 = ₹153,117

However, if the interest rate declared in any year is less than the maximum probable rate then the accumulation of investment in the three year period would be below ₹153,000.

Thus, the probability to declare highest probable rate of interest in each of the years is,

1/3 x 0.4 x 0.9 = 0.12 **[3]**

[Total Marks-8]

<u>Solution 10</u> :

<u>i.</u>

$$NPV_x = -39{,}90{,}000 - 20{,}00{,}000 \times v^2 + \{11{,}20{,}000 \times v^5 + \ 11{,}20{,}000 \times 1.02752 \times v^6 + \cdots \ldots + 11{,}20{,}000 \times 1.02752^{14} \times v^{19}\} @12\%$$

The item in brackets can be solved as:

$$11{,}20{,}000 v^5 (1 + 1.02752 v + \cdots .. + 1.02752^{14} v^{14})$$

$$= 11{,}20{,}000\ v^5 @12\% \times \ddot{a}_{\overline{15}|} @9\% \qquad as\ 1.02752v = \frac{1}{1.09}$$

$$= 11{,}20{,}000 \times 0.56743 \times 8.0607 \times 1.09$$

= 55, 83,796

$$\therefore NPV_x = -39{,}90{,}000 - 20{,}00{,}000 \times 0.79719 + 55{,}83{,}796$$

= - ₹584

[6]

ii. As the amount in (i) above less is than 1000, it is ignored and the NPV of project X is zero. Therefore, IRR=12%

[1]

iii. NPV of project Y is:

$$-2{,}00{,}000\,\ddot{a}_{\overline{12}|} + 120{,}000\,v^{11}\left(Ia_{\overline{15}|}\right) @11\%$$

$$-2{,}00{,}000 \times 7.2065 + 120{,}000 \times 0.3172 \times 44.06 = ₹2{,}35{,}799.84$$ **[4]**

iv. The IRR of project X is 12%. Therefore, to find the value of A, the NPV @12% will become zero.

$$-2{,}00{,}000\,\ddot{a}_{\overline{12}|} + Av^{11}\left(Ia_{\overline{15}|}\right) = 0 @12\%\text{p.a}$$

$$-2{,}00{,}000 \times 6.1944 \times 1.12 + A \times 0.28748 \times 40.7310 = 0$$

$$A = \frac{1387545.60}{40.7310 \times 0.28748}$$

A = ₹1, 18499/-

[3]

[Total Marks-14]

Solution 11 :

i. Present value of the liabilities is:

$100000 \times a_{\overline{12}|} + 300000 \times v^{20}$ at 6%

$= 100000 \times 8.3838 + 300000 \times 0.31180$

= ₹931,920 **[2]**

ii. Discounted Mean Term (DMT) the liabilities are:

$$\frac{(1 \times 100000 \times v + 2 \times 100000 \times v^2 + \ldots\ldots + 12 \times 100000 \times v^{12}) + 300000 \times 20 \times v^{20}}{100000 \times a_{\overline{12}|} + 300000 \times v^{20}}$$

at 6%

$$= \frac{(100000 \times Ia_{\overline{12}|}) + 300000 \times 20 \times v^{20}}{100000 \times a_{\overline{12}|} + 300000 \times v^{20}}$$

$$= \frac{(100000 \times 48.7207) + 300000 \times 20 \times 0.31180}{931{,}920}$$

$$= \frac{6{,}742{,}870}{931{,}920} = 7.2354 \text{ years}$$

[4]

iii. Let the nominal amounts of the Security-X and Security-Y be equal to X and Y respectively:

If the PV of assets is equal to the PV of liabilities then:

$$X\,(0.07\,a_{\overline{7|}} + v^{7}) + Y\,(0.05\,a_{\overline{22|}} + v^{22}) = 931{,}920 \quad \text{at } 6\% \qquad (A)$$

If the DMT of assets is equal to the DMT of liabilities then:

$$\frac{X\,(0.07\,Ia_{\overline{7|}} + 7v^{7}) + Y\,(0.05\,Ia_{\overline{22|}} + 22v^{22})}{931{,}920} = 7.2354$$

i.e. $$X\,(0.07\,Ia_{\overline{7|}} + 7v^{7}) + Y\,(0.05\,Ia_{\overline{22|}} + 22v^{22}) = 6{,}742{,}870 \quad \text{at } 6\% \qquad (B)$$

From (A), we can have:

$$X\,(0.07 \times 5.5824 + 0.66506) + Y\,(0.05 \times 12.0416 + 0.27751) = 931{,}920$$

$$1.055828X + 0.87959Y = 931{,}920$$

From (B), we can have:

$$X\,(0.07 \times 21.0321 + 7 \times 0.66506) + Y\,(0.05 \times 110.9827 + 22 \times 0.27751) = 6{,}742{,}870$$

$$6.127667X + 11.654355Y = 6{,}742{,}870$$

Then,

$$6.127667\left(\frac{931{,}920 - 0.87959Y}{1.055828}\right) + 11.654355Y = 6{,}742{,}870$$

$$Y\left(11.654355 - \frac{6.127667 \times 0.87959}{1.055828}\right) = 6{,}742{,}870 - \frac{6.127667 \times 931{,}920}{1.055828}$$

$$Y\,(6.549513) = 1334322.94$$

Thus, Y = ₹2, 03,728.57

And X = $\frac{9,31{,}920 - 0.87959 \times 2,03{,}728.57}{1.055828}$ = ₹7,12,921.41

[7]

iv. The third condition for immunization is that that convexity of the assets is greater than that of the liabilities. It appears that the asset payments are more spread around DMT than the liability payments, which indicates that the convexity of assets (Security-X and Security-Y) is more than the convexity of liabilities. Hence, the investment trust liabilities are likely to be immunized against small, uniform changes in interest rates.

[2]

[Total Marks-15]

Solution 12 :

i. The monthly effective interest rate is: 0.6434%

The payment reaches ₹ 700 at time t where t is:

100 + 5(t-1) = 700

T = 121 months

The principal of the loan is:

Present value of increasing instalments up to 121 months plus present value of level payment of ₹700 per month for the remaining 59 months is

$$Loan\ principal = 95a_{\neg 121} + 5(Ia)_{\neg 121} + 700a_{\neg 59}v^{121}$$

$$a_{\neg 59} = 48.9643;\ a_{\neg 121} = 83.8926$$

$$(Ia)_{\neg 121} = 4467.52$$

Loan principal = ₹46,082

Alternatively, the principal can be calculated as:

$$Loan\ principal = 95a_{\neg 120} + 5(Ia)_{\neg 120} + 700a_{\neg 60}v^{120}$$

[6]

ii. The Capital outstanding after the payment of the 24th instalment is:

The 25th instalment is: 100+24X5=220

Loan outstanding after the payment of the 24th instalment = P.V of future instalments

Therefore, the capital outstanding after the 24th instalment is:

$$Outstanding\ loan = 215a_{\neg 97} + 5(Ia)_{\neg 97} + 700a_{\neg 59}v^{97}$$

$$a_{\overline{59}} = 48.9643;\ a_{\overline{97}} = 71.9894;\ (Ia)_{\overline{97}} = 3167.86$$

$$Outstanding\ loan = 215 \times 71.9894 + 5 \times 3167.86 + 700 \times 48.9643 \times 0.53681$$

Outstanding loan = ₹49,716.20

Alternatively, the principal can be calculated as:

$$Outstanding\ loan = 215a_{\overline{96}} + 5(Ia)_{\overline{96}} + 700a_{\overline{60}}v^{96}$$

[4]

iii. The outstanding loan at the beginning of the 25th instalment is 49,716.20

The interest paid is 49,716.20X0.006434 = ₹319.87

The capital portion = 220-319.87= - ₹99.87

The Capital due in the 25th instalment is negative because the repayments are increasing overtime.

[3]

iv. The rescheduled instalment will be:

$$Xv + X(1.01923)v^2 + \cdots + X(1.01923)^9v^{10} = 49716.20 \text{ @6\%}$$

$$(1.01923)v \quad @6\% = v @ 4\%$$

The above equation can be rewritten as:

$$\frac{X}{1.01923}a_{\overline{10}}@4\% = 49716.20$$

$$X = 49716.20 \times \frac{1.01923}{8.1109}$$

X = ₹6,247.40 **[4]**

[Total Marks-17]

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