INSTITUTE OF ACTUARIES OF INDIA

Examination – May 2009

Subject: CT1 (Financial Mathematics)

Indicative Solution

General guidelines to markers:

The solutions provided here are indicative ones. Please award appropriate marks for any correct alternative solutions.

Please award marks for correct steps as indicated in the indicative solution even if the final answer does not match exactly.

If data input in a solution is wrong, please do not deduct more than 30% of maximum marks allocated to that part of the question.

Q.1 Let us consider the accumulation of Rs.100 of initial investment.

(i) Term 6 months

Account A:-

Accumulated amount = 100* 6/12 * 0.12 + 100
= 106.00

Account B:-

Accumulated amount = 100*(1.12)^0.5
= 105.83

Account A (Simple interest) gives higher maturity amount.

(ii) Term 12 months

Account A:-

Accumulated amount = 100* 12/12 * 0.12 + 100
= 112.00

Account B:-

Accumulated amount = 100*(1.12)^1
= 112.00

Both Accounts give equal maturity amount.

(iii) Term 18 months

Account A:-

Accumulated amount = 100* 18/12 * 0.12 + 100
= 118.00

Account B:-

Accumulated amount = 100*(1.12)^1.5
= 118.53

Account B (Compound interest) gives higher maturity amount.

[3]

(Note to markers: Even if calculations are not illustrated in the answers, full marks may be awarded if interpretation is correct. The question does not expect calculations to be shown.)

Q.2

(i) Real Rate of interest: Real rate of interest is the rate of interest which will have been earned on a transaction so as to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effects of inflation.

Alternate definition:-This is the interest rate achieved on a transaction allowing for the effect of inflation on the cashflows. The monetary value of the cashflow is adjusted for the inflation during the period.

(ii) Money Rate of interest: Money rate of interest is the rate which will have been earned on a transaction so as to produce the total amount of cash in hand at the end of the accumulation period.

Alternate definition:- This is the interest rate achieved on a transaction calculated ignoring the effects of inflation.

(iii) Comments:

During the period of inflation, monetary rate of interest will be higher.

During the period of deflation (negative inflation), real rate of interest will be higher.

And in periods of zero inflation, both will be equal.

[3]

Q.3

(i) **$0 \le t < 9$**

$$A(0,t) = \exp \int_0^t \{0.09 + 0.0006\, s^2\} ds$$

$$= \exp \{ 0.09t + 0.0006 \frac{t^3}{3} \}$$

$$= \boldsymbol{\exp \{0.09t + 0.0002\, t^3\}} \qquad \textit{(A)}$$

$9 \le t < 15$

$$A(0,t) = \exp\{0.09*9+0.0002*9^3\}\exp \int_9^t \{0.1836 - 0.005\, s\} ds \qquad \{\text{From A}\}$$

$$= \exp\{0.9558\}\exp \{ 0.1836(t-9) - 0.005\frac{(t^2-9^2)}{2}\}$$

$$= \exp\{0.9558\}\exp \{ 0.1836t - 1.6524 - 0.0025t^2 + 0.2025\}$$

$$= \boldsymbol{\exp\{0.1836t - 0.0025t^2 - 0.4941\}} \qquad \textit{(B)}$$

[OR $2.60075\exp\{0.1836t - 1.4499 - 0.0025t^2 \}$]

[OR $0.61012 \exp\{0.1836t - 0.0025t^2\}$]

$t \ge 15$

$$A(0,t) = \exp\{0.1836*15 - 0.0025*15^2 - 0.4941\}\exp \int_{15}^t \{0.1086\} ds \qquad \{\text{From B}\}$$

$$= \exp\{1.6974\}\exp \{ 0.1086(t-15)\}$$

$$= \exp\{1.6974\}\exp \{ 0.1086t - 1.629\}$$

$$= \boldsymbol{\exp\{0.1086t + 0.0684\}}$$

(ii) A(0,17) = exp{0.1086*17 + 0.0684}
= exp{1.9146}
= 6.784225

Accumulation of Rs. 5000 at time 17 = 6.784225 * 5000 = ***Rs. 33921.12***

(iii) Let i be the equivalent effective annual rate over the 17 year period.

Then, (1+i)^17 = A(0,17)
(1+i)^17 = 6.784225 == > i = 6.784225^(1/17)-1
i = 11.9211%

*(**Note**: If a candidate has calculated incorrect expression for A(0,t) for t<9 or t<15, marks may be deducted only for the subdivision where the error has been committed. However, the effect of the erroneous expression on the subsequent parts need not be penalised again provided that sub-division has been attempted correctly.)*

(iv) Let X be the amount of money required to be invested now to get an accumulation of Rs.6000 at time 18.

Then, X *.A(0,18)=6000
Where A(0,18)= exp{0.1086*18 + 0.0684}
= exp{2.0232}
= 7.562486

∴ X = 6000/7.562486 = 793.3899
X = 793.39

[12]

Q.4

(i) Given i=.08.

$[1- d^{(12)}/12]^{12} = 1\text{-}d = v$

=> $d^{(12)} = 12\ [1 – v1^{/12}]$

=> $d^{(12)}$ = 12*[1-1.08^(-1/12)]= 0.076714776

$[1+ i^{(365)}/365]^{365} = [1+i]$

$i^{(365)}$ = 365*[1.08^(1/365)-1]= 0.076969155

δ = ln(1+i) = ln(1.08) = 0.076961041

$i^{(1/2)}$ = 0.5* (1.08 ^ 2 – 1) = 0.0832

(ii) **Method I : Using Present Values**

Let PV of payments made in first 3 years = X
Note that there is no payment in first year. There is a payment of 300 p.a. in second year and 400 p.a. in third year payable quarterly.
Effective rate of interest p.a. for first 3 years = (1+.08/12) ^12 - 1
= 0.082999507

$X = 0 + v\, 300 a_{\overline{1|}}^{(4)} + v^2\, 400\, a_{\overline{1|}}^{(4)}\ _{@\, 8.2999507\%}$

$a_{\overline{1|}}^{(4)} = 0.951623562\ _{@\, 8.2999507\%}$

X =0.951623562(300v + 400v2)
=0.951623562 * 618.0469868 = 588.1480748

X = 588.1481

Let PV of payments made in second 3 years be = Y
Effective rate of interest p.a. for second 3 years = $(1+.12/2)^2 - 1$
= 0.1236

$Y = v^3{}_{@\, 8.2999507\%}(500 a_{\overline{1|}}^{(4)} + 600 v\, a_{\overline{1|}}^{(4)} + 700 v^2 a_{\overline{1|}}^{(4)}\ _{@\, 12.36\%})$

$a_{\overline{1|}}^{(4)} = 0.9302464865\ _{@\, 12.36\%}$

Y =0.7872546299*0.9302464865*(500+600v+700v2)$_{@\, 12.36\%}$
= 1163.296663

Y = 1163.2967

Corpus = 10,000

X + Y = 588.1481+1163.2967 = 1751.4448

Amount available at
the end of 6 years = $(10{,}000-1751.4448)*1.08299^3 *1.1236^3$
=14862.7050

[12]

Method II : Using Accumulation

Let Accumulation at the end of 6^{th} years of payments made in first 3 years be A.
There is no payment in the first year.

$A = [\,(1+i)300 S_{\overline{1|}}^{(4)} + 400 S_{\overline{1|}}^{(4)}\ _{@\, 8.299950681\%}]\,(1.1236)^3$

$S_{\overline{1|}}^{(4)}\ _{@\, 8.299950681\%} = 1.030607848$

$A = 1.030607848\,[300(1+i)+400]*1.1236^3 = 1059.757864$

A = 1059.7579

Let Accumulation of Payments made in second 3 years at the end of 6th year be B.

$B = [\ 500 s_{\overline{1}|}^{(4)} (1+i)^2 + 600 s_{\overline{1}|}^{(4)} (1+i) + 700\ s_{\overline{1}|}^{(4)}\ ]$ @ 12.36%

$s_{\overline{1}|}^{(4)}{}_{@\ 12.36\%} = 1.045224952$

$B = [500(1+i)^2 + 600(1+i) + 700]*1.045224952 = 2096.092531$

B = 2096.0925

Accumulation of corpus = $10{,}000(1.082999507)^3(1.1236)^3$

= ***18018.5554***

Amount available at
the end of 6 years = 18018.5554 – A – B
= 18018.5554 – 1059.7579 – 2096.0925
= 14862.7050

[8]

Method III : Using Accumulation but different effective interest rates :

The accumulated value of Rs. 10,000 at the end of 6 years

$= 10{,}000\ (1+.08/12)^{36} *(1+.12/2)^6$

$= 18{,}018.56$

During first 3 years, given $i^{(12)} = 0.08$

⇨ $i^{(12)}/12 = 0.006667$

Therefore, $i^{(4)}/4 = (1.006667)^3 - 1 = 0.02013$

Therefore, the accumulated value of payments made during first 3 years at the end of 3rd year

$= 75\ s_{\overline{8}|\ @0.02013} + 25\ s_{\overline{4}|\ @0.02013}$

(There is no payment in the first year. During 2nd year a payment of Rs. 75 is made at the end of each quarter. During the 3rd year payment of Rs 100 is made at the end of each quarter, which can be rewritten as 75 + 25)

= 75 (8.587023) + 25 (4.122431)
= 747.0874 -------- (A)

During last 3 years, given $i^{(2)} = 0.12$

⇨ $i^{(2)}/2 = 0.06$

Therefore, the accumulated value of all payments at the end of 6th year

$= (A)*(1.06)^6 + 250\, s^{(2)}_{\overline{6|}\, @0.06} + 50\, s^{(2)}_{\overline{4|}\, @0.06} + 50\, s^{(2)}_{\overline{2|}\, @0.06}$

$\{s^{(2)}_{\overline{6|}\, @0.06} = 7.0784243;\ s^{(2)}_{\overline{4|}\, @0.06} = 4.439279;\ s^{(2)}_{\overline{2|}\, @0.06} = 2.090449905\}$

= Rs. 3155.85

Therefore, net amount remaining in the corpus at the end of 6 years = 18018.56 – 3155.85

= ***Rs. 14,862.7050***

[8]

Q.5

(i) ***Money weighted rate of return***

Let i be the MWRR p.a. over the three year period. Then,

$45283(1+i)^3 + 4500(1+i)^2 + 3247(1+i) + 1321 - 2884(1+i)^{1/2} = 63677$

⇨ First trial using binomial approximation :-

⇨ 45283(1+3i) + 4500(1+2i) + 3247(1+i) + 1321-2884(1+.5i) = 63677

⇨ 51467+i(146654)=63677

⇨ i = 8.33% approximately

⇨ At i=8% LHS = 64122.96

⇨ At i=7% LHS = 62437.73

⇨ Interpolating between 7% and 8%, **i=7.735%**

⇨ ***MWRR = 7.74 %*** (nearest 0.01%)

(ii) ***Time Weighted rate of return***

TWRR is found from the equation

$$(1+i)^3 = \frac{(48492 - 4500)}{45{,}283} \cdot \frac{(52706 - 3247)}{48{,}492} \cdot \frac{(59251 + 2884)}{52706} \cdot \frac{(63677-1321)}{59251}$$

= 0.971490405 * 1.019941434 * 1.178898038*1.052404179

= 1.229341546

i = 1.229341546^(1/3)-1 = 7.1250%

⇨ ***TWRR = 7.13%*** (nearest 0.01%)

(iii) The TWRR is considered a better measure of fund performance as it eliminates the effects of cashflow amounts and timings. MWRR is sensitive to the amounts and timings of the net cashflows.

[9]

Q.6

(i)

1) Government bonds are issued by governments. Unsecured Loan Stocks are issued by companies.
2) Government bonds are usually regarded as the most secure type of debt (especially if issued by governments of developed countries). Unsecured Loan Stocks are less secure as they are not secured on assets of the company.
3) Government bonds are more marketable than Unsecured Loan Stocks. This is because they tend to be issued in large volumes.
4) The yields on government bonds are lower than on equivalent Unsecured Loan Stocks, mainly due to greater security and higher marketability of government bonds.

(ii) For determining price paid by Investor A ignore tax as yield given is gross redemption yield.

Let Price paid by Investor A be P1.

Then $P1 = 4.5\, a_{\overline{8}|}^{(2)} + 110\, v^8$ @ 6.25%

= 4.5 * 6.243434 + 110 * 0.61569906
= 28.095454 + 67.726897
= 95.822351

Price paid by Investor A = ***Rs. 95.82*** per 100 nominal

(iii) For determining price paid by Investor B, we need to find out if the deal is subject to capital gains tax or not.

If $i^{(p)} > (1-t)g$ then there is capital gain.

i.e., If $i^{(2)} > (1-0.25)0.045$
{where $i^{(2)} = 2*(1.06\hat{}(1/2)-1) = 0.059126028$}

0.059126028 > (1-.25)*0.045=0.03375

Hence, $i^{(2)} > g(1-t_1)$

Hence there is capital gain on the contract.

Let Price to be paid by Investor B = P2

Then, $P2 = 4.5*(1-0.25)\, a_{\overline{6}|}^{(2)} + 110\, v^6 - 0.32(110-P2)\, v^6 @ 6\%$

$= 3.375*4.99001 + 77.545659 - 24.814611 + 0.225587\, P2$

⇨ $P2 = 69.572332/ 0.774413$

⇨ $P2 = 89.8388$

⇨ Price paid by Investor B = ***Rs. 89.84*** per 100 nominal.

[12]

Q.7

i) 3 types of derivatives are Options, Futures and Swaps.

ii) Arbitrage is a risk free trading profit.
An arbitrage opportunity exists if either:

a) an investor can make a deal that would give him an immediate profit, with no risk of future loss, or

b) an investor can make a deal that has no initial cost, no risk of future loss, and a non-zero probability of a future profit.

In the major developed securities market arbitrage opportunities, when they do arise are very quickly eliminated as investors spot them and trade on them. Such opportunities are so fleeting in nature, according to empirical evidence, that it is sensible and realistic to assume that they do not exist.

iii) Let K be the forward price.
Price of share, $S_o = 120$

Using formula, $K = S_o e^{\delta T} = 120\, e^{\delta * 3/12}$

$= 120\,(1 + i^{(4)}/4) = 120\,(1.015)$

= ***Rs. 121.80***

[7]

Q.8 Working in crores;

PV of outlay as at 01.01.2008, A (say)

$= 1000 + 750(v^{0.5} + v^{11/12}) + 5*12v\, \ddot{a}_{\overline{1.5}|}^{(12)}$ @ 10% $= 2478.8932$

⇨ ***A= 2478.8932***

PV of ADF income as at 01.01.2008, B (say)

$= \{1.5*200\, \bar{a}_{\overline{1}|} + 1.5*200*1.05\, \bar{a}_{\overline{1}|}\, (v + v^2)$

$+ 0.5*1000\, \bar{a}_{\overline{1}|} + 0.5*1000*1.2\, \bar{a}_{\overline{1}|}\, (v + v^2)\} * v^{14/12}$ @ 10%

$= \{800\, \bar{a}_{\overline{1}|} + 915\, \bar{a}_{\overline{1}|}\, (v + v^2)\} * v^{14/12}$

[OR $= \{800\, \bar{a}_{\overline{3}|} + 115\, v\, \bar{a}_{\overline{2}|}\} * v^{14/12}$

$= \{800\, a_{\overline{3}|} + 115\, v\, a_{\overline{2}|}\} * i/\delta * v^{14/12}$ ***]***

Using the actuarial tables,

$a_{\overline{3}|} = 2.4869$ $a_{\overline{2}|} = 1.7355$ $v = 0.90909$ $i/\delta = 1.049206$

⇨ ***B = 2038.075673***

PV of revenue from parking as at 01.01.2008, C (say)

$= \{10 + 11v + 12\, v^2 + 13\, v^3 + \ldots\ldots\ldots\ldots + 18\, v^8\} * v^{32/12}$

$= \{10\, \ddot{a}_{\overline{9|}} + 1\, (Ia)_{\overline{8|}}\} * v^{32/12}$

$= \{10\, (6.3349) + 21.3636\} *0.775565 = 65.70015$

⇨ $C = \boldsymbol{65.70015}$

PV of Rent from duty free shops as at 01.01.2008, D (say)

$= 12v^3\, \{\ddot{a}_{\overline{1|}}^{(12)} + 1.05v\, \ddot{a}_{\overline{1|}}^{(12)} + (1.05)^2v^2\, \ddot{a}_{\overline{1|}}^{(12)} + \ldots\ldots\ldots + (1.05)^9v^9\, \ddot{a}_{\overline{1|}}^{(12)}\}$

$= 12v^3\, \ddot{a}_{\overline{1|}}^{(12)}\, \{1 + 1.05v + (1.05)^2v^2 + \ldots\ldots\ldots + (1.05)^9v^9\}$

$= 12v^3\, \ddot{a}_{\overline{1|}}^{(12)}\, \{\ddot{a}_{\overline{10|}}\ @\, j\}$, where $1/(1+j) = 1.05/1.1 \Rightarrow j = 0.047619$,

$\ddot{a}_{\overline{10|}}\ @\, j = 8.183867$

⇨ $\boldsymbol{D = 70.6570}$

PV of all income $= B + C + D$

$= 2038.075673 + 65.7001 + 70.657$

$= \boldsymbol{2174.4327}$

At 10% risk discount rate, B+C+D < A

i.e., PV of all income (2174.43) < PV of all outlays (2478.89)

Hence the project is not viable. **[17]**

Q.9

(i) $1+f_{1,1} = \dfrac{1.075^2}{1.065} = 1.085093$

$1+f_{2,1} = \dfrac{1.08^3}{1.075^2} = 1.090070$

$1+f_{3,1} = \dfrac{1.0825^4}{1.08^3} = 1.090035$

P.V. of annuity a year from now

$$= 5000 * \left\{ \frac{1}{(1+f_{1,1})} + \frac{1}{(1+f_{1,1})(1+f_{2,1})} + \frac{1}{(1+f_{1,1})(1+f_{2,1})(1+f_{3,1}))} \right\}$$

$= 5000 * \{0.92157923+0.845431337+0.775600334\}$

$= 4607.896 + 4227.157 + 3878.002$

= 12713.05452

(ii) Redington's theory of immunization states that the surplus in a fund an be immunized to small changes in interest rates if :-

1. Value of assets at the starting rate of interest is equal to the value of liabilities. i.e., PV of assets = PV of liabilities
2. Volatility of assets = Volatility of Liabilities OR DMT(Assets)=DMT(Liabilities)
3. Convexity of Assets > Convexity of liabilities

(iii)

(a) PV of Bond = $\Sigma C_t v^t = 16\, a_{\overline{8}|}^{(2)} + 100 v^8$ @ 7% p.a.

$= 16 * 6.074028855+58.20091046$

$= 155.3853721$

$$\text{Duration} = \frac{\Sigma\, t\, C_t v^t}{\Sigma C_t v^t}$$

Working in half years,

Effective rate of interest per half year = 1.07^0.5-1 = 3.4408%

$\Sigma\, t\, C_t v^t = 8(1v+2v^2+3v^3+4v^4+\ldots\ 16v^{16}) + 16*100v^{16}$ @ 3.4408%

$= 8(Ia)_{\overline{16}|} + 1600v^{16}$

$= 8*94.56808996 + 931.2145673$

$= 1687.759287$

Duration = 1687.759287 /155.39 = 10.86 half years

= ***5.43072 years***

(b) The duration would become slightly shorter, because the coupon rates are now about 25% higher than earlier, which masks marginally the effect of the redemption cashflow at the end of 8 years as also the weightage of the terms < n are more now.

[17]

Q.10

The required accumulated value of the investments is given by:

$500(1+i_2)(1+i_3) + 1000\,(1+i_3) + 1500 = X$ say.

We are required to find E(X) and V(X).

$E(X) = E[500(1+i_2)(1+i_3) + 1000\,(1+i_3) + 1500]$

As i_t's are independent, this can be simplified as follows:

$$\begin{aligned} E(X) &= 500E(1+i_2)E(1+i_3) + 1000E(1+i_3) + 1500 \\ &= 500(1.07)(1.085)+1000(1.085)+1500 \\ &= 580.475 + 1085 + 1500 = 3165.475 \end{aligned}$$

Mean of Accumulated Value = 3165.48

To find the variance of X we can use the fact that Var(X+k) = Var(X) where k is a constant.

$$\begin{aligned} V(X) &= V\left\{500(1+i_2)(1+i_3) + 1000\,(1+i_3) + 1500\right\} \\ &= V\left\{500(1+i_2)(1+i_3) + 1000\,(1+i_3)\right\} \qquad \text{since } V(k)=0 \\ &= 500^2 V\left\{(1+i_3)(1+i_2+2)\right\} \\ &= 500^2 V\left\{(1+i_3)(3+i_2)\right\} \end{aligned}$$

{Now, we can use the formula $V(Y) = E(Y^2) - E^2(Y)$}

$$\begin{aligned} V\left\{(1+i_3)(3+i_2)\right\} &= E\left\{(1+i_3)^2(3+i_2)^2\right\} - E\left\{(1+i_3)(3+i_2)\right\}^2 \\ &= E\left\{(1+i_3)^2\,(3+i_2)^2\right\} - E^2\left\{(1+i_3)(3+i_2)\right\} \\ &= E(1+i_3)^2\,E(3+i_2)^2 - \{(1+E(i_3))(3+E(i_2))\}^2 \\ &= \{V(1+i_3) + E(1+i_3)^2\}.\{V(3+i_2) + E(3+i_2)^2\} - \{1.085\text{x}3.07\}^2 \\ &= \{.01^2 + 1.085^2\}.\{.005^2 + 3.07^2\} - \{1.085\text{x}3.07\}^2 \\ &= 0.000971923125 \end{aligned}$$

Therefore, $V(X) = 500^2 \text{ x } .000971923125 = 242.9807813$

$\text{S.D.}(X) = 500 * \text{S.D}\left\{(1+i_3)(3+i_2)\right\} = 15.58784081$

Std Deviation of Accumulated Value = 15.59

[8]

[Total 100 Marks]

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