Actuarial Society of India

Examinations

May 2006

CT8 – Financial Economics

Indicative Solutions

Sol 1 (i)(a) Risk-neutral models

? Risk-neutral models are a class of no arbitrage model, which are mathematical modelsbuilt on the assumption that investment markets are arbitrage-free. They are typically used to price derivatives and other securities that contain options.
? In a risk-neutral model, the probabilities are adjusted so that all assets (risky and riskfree) yield an expected return equal to the risk-free rate of return.
? This is consistent with a world in which investors are in fact risk-neutral or indifferent to risk, and therefore base their investment choices solely on expected return. Expected returns are adjusted by a change of the probability measure describing the probability of different price movements.
? The cash flows then obtained in the risk-neutral world are discounted at the risk-free rate to obtain the current fair price of the asset.

(4)

Sol 1 (i)(b) Equilibrium models

? Equilibrium models are based on a simplified model of an investment market, or in the case of general equilibrium models, a simplified model of the economy as a whole.
? They are built on the assumption that a large number of identical investors seek to maximise their individual expected utilities by suitable investment choices, according to a particular utility function and subject to appropriate constraints.
? The models are set up and solved, and the results obtained then characterise the equilibrium position of the market, in which all investors have maximised their expected utilities and so have no wish to change their investment allocations.
? In particular, the models indicate the expected investment returns and prices that apply when the market is in its equilibrium state.

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(4)

Sol 1 (ii) Advantages and disadvantages of different approaches
Risk-neutral models:

? may be easier to use to price options and other derivatives
? may reproduce actual market prices and yields more accurately, particularly for options.

Equilibrium models:

? are based on economic theory and therefore the results obtained have an economic interpretation, whereas risk-neutral models are purely mathematical
? can – in the case of general equilibrium models – model the whole economy and not just a particular investment market
? may be used to price securities when risk-neutral models cannot be applied – *eg* if the market is incomplete and so a replicating portfolio cannot be constructed.

(4)

In practice, many models (such as Black-Scholes) can be derived using either the risk neutral or (general) equilibrium approaches.

[Total 12]

Sol 2(i) Expected option payoff

Given the strike price of Rs 0.50, the payoff is Rs 0.50 if the share price goes up, otherwise it is zero. So:

$$E[C_1] = 0.6 \times 0.5 + 0.4 \times 0 = £0.30$$

(2)

Sol 2 (ii) Replicating portfolio

If we let f be the number of shares held at time 0 and y be the amount of cash, then for replication we require:

$$11\phi + e^{0.04/12}\psi = 0.50$$

$$9.5\phi + e^{0.04/12}\psi = 0$$

Solving these equations gives:

$$\phi = 1/3 = 0.333$$

and $\psi = -9.5 \times \frac{1}{3} \times e^{-0.04/12} = -3.156$

The fair price is therefore:

$$10\phi + \psi = £0.18$$

(4)

Sol 2

(iii) ***Number of options to construct risk-free portfolio***

Since one derivative can be replicated by ϕ shares and ψ cash, as we saw in part (ii), one derivative and $-\phi$ shares, or equivalently, $-\frac{1}{\phi}$ derivatives and one share must replicate the cash, *ie* they will form a risk-free portfolio.

We therefore need to sell $\frac{1}{\phi} = 3$ call options for every share owned to construct a risk-free portfolio.

(2)

Sol 2

(iv) ***Arbitrage opportunity***

The discounted value of the expected payoff is $0.3e^{-0.04/12} = £0.29900$. [½]

This is greater than the fair price found in Part (ii), so we need to *sell* the options. [½]

By part (iii), a portfolio consisting of 1 share and -3 options will be risk-free. [½]

This portfolio will cost $10 - 3 \times 0.29900 = £9.103$. [1]

We borrow this amount in the cash market to have zero initial expenditure. [½]

At the end of the month you would owe $e^{0.04/12} \times 9.102995 = £9.13$ … [1]

… but have a portfolio worth either: [1]

$11 - 3 \times 0.5 = £9.50$ if the share price goes up [½]

or $9.50 - 3 \times 0 = £9.50$ if the share price goes down. Rs 0.37 per share [½]

Note the portfolio is constructed to be risk-free and hence the payoffs must be equal.

Either way, you make a (risk-free) profit of $9.50 - 9.13 = £0.37$ per share. [½]

(Max 5)

Sol 2)

(v) ***Risk-neutral valuation of option***

The general risk-neutral valuation formula for the price at time t of a derivative that pays X at time T is:

$$V_t = e^{-r(T-t)} E_Q\left[X \middle| F_t \right]$$ [½]

where r is the risk-free force of interest and Q is the risk-neutral measure.

Here the tree is recombining and:

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.04/12} - 0.95}{1.1 - 0.95} = 0.35559$$ [½]

There are 5 possible share prices at time 4:

(1) 14.641 [¼]

(2) 12.6445 [¼]

(3) 10.92025 [¼]

(4) 9.431125 [¼]

(5) 8.1450625 [¼]

These occur with risk-neutral probabilities $\binom{4}{i} q^i (1-q)^{4-i}$: [¼]

(1) $p_1 = 0.015989$ [¼]

(2) $p_2 = 0.11590$ [¼]

(3) $p_3 = 0.31505$ [¼]

(4) $p_4 = 0.380622$ [¼]

(5) $p_5 = 0.172442$ [¼]

The corresponding payoffs for the derivative are:

(1) $X_1 = 2 \times (14.641 - 12) = 5.282$ (exercise and buy 2 shares at £12) [¼]

(2) $X_2 = 2 \times (12.6445 - 12) = 1.289$ (exercise and buy 2 shares at £12) [¼]

(3) $X_3 = 0$ (do not exercise) [¼]

(4) $X_4 = 0$ (do not exercise) [¼]

(5) $X_5 = 1 \times (9 - 8.145063) = 0.854937$ (exercise and sell 1 share at £9) [¼]

It follows that the fair price of the derivative is:

$$V_0 = e^{-4 \times 0.04/12} \left(\sum_{i=1}^{5} X_i q_i \right) =$$

Rs 0.38 *[1]*

(Max 5)

Sol 3)a)

Single Factor Vasicek

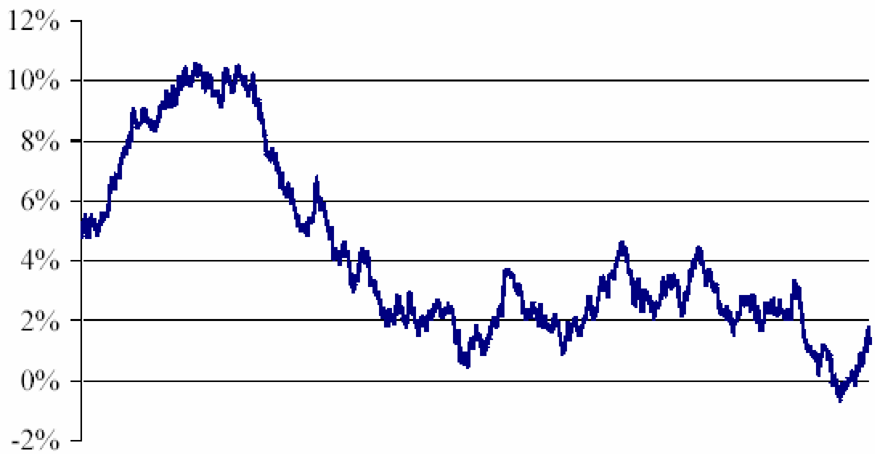
The Vasicek model has the dynamics, under Q:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma d\tilde{W}(t)$$

where $\tilde{W}(t)$ is a standard Brownian motion under Q.

The parameter α takes a positive value.

The graph below show a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.02$.



The solution to this equation can be expressed as:

$$r(t) = r(0)e^{-\alpha t} + \mu\left(1 - e^{-\alpha t}\right) + \sigma\int_0^t e^{-\alpha(t-u)}d\tilde{W}_u$$

This SDE tells us about the changes in the short rate over each instant.

The drift coefficient can be written as $-\alpha[r(t) - \mu]$. So the drift depends on the current interest rate. Since $\alpha > 0$, the sign of this coefficient always directs the movements towards μ. So the process is mean-reverting towards the constant mean value μ.

Since the random component of the movements is based on Brownian motion, the movements in the short rate are normally distributed. The volatility parameter σ specifies how big the random movements are.

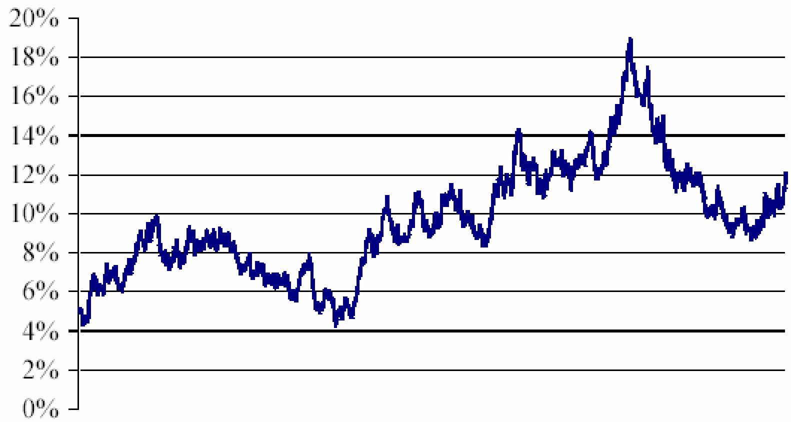
(2)

Sol 3)b) Cox-Ingersoll-Ross

The Cox-Ingersoll-Ross model ensures that all interest rates remain positive, thereby countering one of the main drawbacks of the Vasicek model. The SDE for $r(t)$ under Q is:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}(t)$$

The graph below show a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.1$.



Prices of zero-coupon bonds are given by:

$$B(t,T) = e^{a(\tau) - b(\tau)r(t)}$$

This equation is the same as for the Vasicek model. As before, $\tau = T - t$.

But now:

- $$b(\tau) = \frac{2(e^{\theta\tau} - 1)}{(\theta + \alpha)(e^{\theta\tau} - 1) + 2\theta}$$
- $$a(\tau) = \frac{2\alpha\mu}{\sigma^2}\log\left(\frac{2\theta e^{(\theta+\alpha)\tau/2}}{(\theta + \alpha)(e^{\theta\tau} - 1) + 2\theta}\right)$$
- $\theta = \sqrt{\alpha^2 + 2\sigma^2}$.

Pricing formulae for European call and put options on zero-coupon bonds look similar to those for the Vasicek model and to the Black-Scholes formulae for equity options. However, where the latter models use the cumulative distribution function of the Normal distribution, the CIR formulae use the cumulative distribution function of the non-central chi-squared distribution.

(4)

Sol 3) c) Hull & White

A simple way to get theoretical prices to match observed market prices is to introduce some elements of *time-inhomogeneity* into the model. The Hull & White (HW) model does this by extending the Vasicek model in a simple way.
We define the SDE for $r(t)$ under Q as follows

$$dr(t) = \alpha\left(\mu(t) - r(t)\right)dt + \sigma d\tilde{W}(t)$$

where $\mu(t)$ is a deterministic function of t. $\mu(t)$ has the natural interpretation of being the local mean-reversion level for $r(t)$.

This is the same equation as for the Vasicek model, except that μ is no longer a constant, but can vary over time.

The Hull & White model can be easily extended to include a time-varying but deterministic $\sigma(t)$. This allows us to calibrate the model to traded option prices as well as zero-coupon bond prices.

(2)

Sol 3)d) Key differences & similarities

These three models are all one-factor models used for modelling the short rate of interest $r(t)$. [½]

All three models assume that $r(t)$ has the dynamics of an Itó process under the risk-neutral probability measure Q. [½]

The equations defining the three models are:

Vasicek:	$dr(t) = \alpha[\mu - r(t)]dt + \sigma d\tilde{W}(t)$	[1]
Cox-Ingersoll-Ross:	$dr(t) = \alpha[\mu - r(t)]dt + \sigma\sqrt{r(t)}d\tilde{W}(t)$	[1]
Hull & White:	$dr(t) = \alpha[\mu(t) - r(t)]dt + \sigma d\tilde{W}(t)$	[1]

All three models are mean reverting to the value μ. [½]

μ is time-dependent for the Hull & White model, but constant for the other two. [½]

The Cox-Ingersoll-Ross model includes the factor $\sqrt{r(t)}$ in the volatility coefficient. This prevents $r(t)$ taking negative values. [½]

The Vasicek model is much more tractable mathematically than the other two. [½]

(6)
[16]

Sol 4)

The force of inflation is modelled as:

$$I(t) = QMU + QA\left[I(t-1) - QMU\right] + QSD.QZ(t)$$

With the relevant parameter values substituted in, the expression for next year's force of inflation is:

$$I(t+1) = 0.035 + 0.7\left[0.02 - 0.035\right] + 0.02.QZ(t+1)$$

ie $I(t+1) = 0.0245 + 0.02.QZ(t+1)$ [2]

From the Tables, the two-sided 95% confidence interval for a standard normal variable is equal to $(-1.96, 1.96)$. Hence, the 95% confidence interval for next year's force of inflation is given by:

$$(0.0245 - 0.02\times1.96, 0.0245 + 0.02\times1.96)$$ [1]

which simplifies to:

$$(-0.0147, 0.0637)$$

ie $(-1.47\%\ pa, 6.37\%\ pa)$ [1]

(4)

Sol 4)

(ii) ***Comment***

The confidence interval is centred on an inflation rate of 2.45% *pa*. This is greater than the current inflation rate of 2% *pa* because the process for the force of inflation is assumed to revert back to the long-run mean value of 3.5% *pa*. [½]

The width of the 95% confidence interval is just over 8%. This seems very large and reflects the value of the *QSD* parameter, which is equal to 2%. [½]

The Wilkie model is designed to simulate possible distributions of inflation rates over long periods of time, rather than for short-term forecasting in the manner of an econometric model. Hence, it may be inappropriate to use it to determine a confidence interval for next year's inflation. *ie* we are using an estimate of *QSD* based on historical data when we really need a cross-sectional estimate, which will have different statistical properties (and will presumably be smaller). [1]

In addition, the quoted parameter values are based upon one investor's judgments and historical data over the last twenty years. They may therefore be inappropriate as the basis for a projection over the coming year due to random variation. [1]

Finally, the Wilkie model residuals are assumed to be normally, and hence symmetrically, distributed. If in practice inflation is not symmetrically distributed about its long-run mean value, then it may be inappropriate to use a symmetrical confidence interval. In practice we might expect inflation to be positively skewed as it is rarely negative. [1]

[Max 3]

Sol 5)

In an *efficient* security market the price of every security fully reflects all available and relevant information. The *efficient markets hypothesis* states that security markets *are* efficient.

Three forms of the efficient markets hypothesis are commonly distinguished:

1. *Strong form* – market prices incorporate all information, whether or not it is publicly available. If markets are strong form efficient, then insider trading cannot be used to generate excess risk-adjusted returns.

2, *Semi-strong form* – market prices incorporate all publicly available information. If markets are semi-strong form efficient, then fundamental analysis cannot be used to generate excess risk-adjusted returns.

3. *Weak form* – market prices incorporate all of the information contained in historical price data. If markets are weak form efficient, then technical analysis cannot be used to generate excess risk-adjusted returns.

[12]

Sol 6)

Assumptions of Black-Scholes

. the underlying share price follows geometric Brownian motion
. the market is complete
. the market is arbitrage-free
. the risk-free rate r is constant and the same for all borrowing and lending
. assets may be bought and sold at any time $t > 0$
. assets may be held in any amount
. there are no taxes or transaction costs

It is clear that each of these assumptions is unrealistic to some degree: for example, Share prices can jump. This invalidates assumption 1 since geometric Brownian motion has continuous sample paths.

An important consequence of discontinuous share prices is that it is not possible to rebalance the risk-free portfolio at each moment so as to eliminate movements in the value of the portfolio. Hence, the portfolio is not entirely risk-free. **However, hedging strategies can still be constructed which substantially reduce the level of risk. The risk-free rate of interest does vary and in an unpredictable way.** We might, for example, assume that the risk-free rate is either the base rate set by the central bank or the yield on Treasury bills, both of which can vary over time.

However, over the short term of a typical derivative the assumption of a constant risk-free rate of interest is not far from reality. (More specifically the model can be

adapted in a simple way to allow for a stochastic risk free rate, provided this is a predictable process.) In addition, different rates may apply for borrowing and lending.

Unlimited short selling may not be allowed except perhaps at penal rates of interest. These problems can be mitigated by holding mixtures of derivatives which reduce the need for short selling.

Shares can normally only be dealt in integer multiples of one unit, not continuously and dealings attract transaction costs.
Transactions costs do arise in practice, their impact depending upon their size.
Distributions of share returns tend to have fatter tails than suggested by the log-normal model

[12]

Sol 7 (i)
Variances

To calculate the variances according to the single-index model, we need estimates of the "error term" variances, $V_{\varepsilon W}$ and $V_{\varepsilon Z}$. These can be obtained as the variance of the residuals in a linear regression of security returns on market returns:

$$V_{\varepsilon W} = \frac{1}{12}\left(S_{WW} - \frac{(S_{WM})^2}{S_{MM}} \right)$$

where:

$$S_{xy} = \sum_{i=1}^{12} (x_i - \overline{x})(y_i - \overline{y}) \qquad [1]$$

$$\Rightarrow \qquad V_{\varepsilon W} = V_W - \frac{(C_{WM})^2}{V_M} = 21.743 - \frac{20.319^2}{39.472} = 11.283 \qquad [1½]$$

Similarly:

$$V_{\varepsilon Z} = V_Z - \frac{(C_{ZM})^2}{V_M} = 41.521 - \frac{22.292^2}{39.472} = 28.952 \qquad [1½]$$

Then the variances according to the single-index model are:

$$V_W = \beta_W^2 V_M + V_{\varepsilon W} = 0.5147^2 \times 39.472 + 11.283 = 21.743 \qquad [1]$$

$$V_Z = \beta_Z^2 V_M + V_{\varepsilon Z} = 0.5647^2 \times 39.472 + 28.952 = 41.521 \qquad [1]$$

Covariance

The covariance is given by:

$$C_{WZ} = \beta_W \beta_Z V_M = 0.5148 \times 0.5647 \times 39.472 = 11.473 \quad [1]$$

(7)

Sol 7(ii)

Under the single-index model, portfolio alphas and betas are simply weighted averages of those of the constituent securities. Hence for a portfolio P:

$$R_P = \tfrac{1}{2}(\alpha_W + \alpha_Z) + \tfrac{1}{2}(\beta_W + \beta_Z)R_M + \tfrac{1}{2}(\varepsilon_W + \varepsilon_Z) \quad [\tfrac{1}{2}]$$

so that:

$$E[R_P] = \tfrac{1}{2}(\overline{w} + \overline{z}) = 5/6 = 0.8333 \quad [1]$$

Hence it is identical to that calculated purely on the basis of historical data.

The variance of P is given by:

$$V_P = \beta_P^2 V_M + \tfrac{1}{4}\left(V_{\varepsilon W} + V_{\varepsilon Z}\right)$$

$$= 0.5398^2 \times 39.472 + \tfrac{1}{4}(11.283 + 29.529) = 21.56 \quad [1\tfrac{1}{2}]$$

This compares to a variance based purely upon historical data of V = 20.381, which is calculated simply by averaging the individual monthly returns for W and Z and then calculating the resulting variance in the usual way.

The two variances differ because the variance calculated under the single-index model depends upon the covariance under the model, which is itself different from the covariance based purely upon the historical data.

(6)

[13]

Sol 8) (a) Explain what is meant by the Wilkie model

The Wilkie model attempts to model the processes generating investment returns for several different types of asset.

It can therefore be used to simulate the possible future development of investment returns, eg as part of an asset liability modeling exercise.

Although it is primarily statistically-based- having been estimated using historical data for the relevant time series involved - it does include some constraints upon the parameter values. These are based upon economic theory and consequently has some features of an

econometric model. This is modeled as a first-order autoregressive process with normally distributed Innovations...
The key variable is the force of inflation that is assumed to be the driving force behind the other variables:
-log of the equity dividend yield
-annual change in the log of dividend income
-log of the real yield on index-linked bonds. .
.
It is therefore a particular case of a vector autoregressive moving average (VARMA) model.
It has a cascade structure so that the process driving each individual variable can be analysed using transfer functions and without the need to consider all the other variables in the model.

(6)

Sol 8 (b) Explain how the models differ
Whereas the lognormal model is entirely statistically-based, having been developed in in response to studies of historical time series data concerning asset returns, the Wilkie Model imposes some constraints on the possible parameter-values to reflect economic Theory. It is therefore partly econometric.
The lognormal model assumes that investment returns in non-overlapping intervals are Independent and that the expected return does not change over time. In contrast, the Wilkie model models key investment variables- e.g. yields- as autoregressive processes That tends to revert to a long-term mean value.
The Wilkie model models the yields and/or prices' produced by several different classes of asset (including equities, conventional and index-linked bonds, and property), whereas the lognormal model is usually applied only to equity prices.
The Wilkie model calculates equity prices indirectly using the equation:
Equity price = equity dividend income divided by the Equity dividend yield rather than generating it directly.

(6)
[12]