Actuarial Society of India

Examinations

May 2006

CT1 –Financial Mathematics

Indicative Solutions

Q1)

(i) Present value of 1 payable at the end of 10 year

=v^10

= Exp[- ($_0$?^{10}(0.004t + 0.0002t^2)dt)]

=Exp [-(0.004 t^2 /2 + 0.0002 t^3 /3)$_0^{10}$]

=Exp [-0.2 - 0.06667] =Exp[-26667]

= 0.765926

Present Value of 1000 payable at the end of 10 year is

= 765.926 **(3)**

(ii) Annual effective rate of interest i is such that

(1+i)^(-10) = 0.765926

(1+i)^(10) = 1.305609

I = 2.70% **(3)**

[6]

Q2)

(i) d is the interest payable at the beginning of the year on a loan of 1unit repayable at the end of the year. The corresponding interest payable at the end of year on 1unit of loan is i. Thus, d must equal the present value at the beginning of the year of the payment i payable at the end of the year ie d= iv .

(2)

(ii) Let d$^{(2)}$ be the nominal rate of discount per annum convertible half yearly

(a) 1-d = (1-0.0225)^4 = 0.912992

d = 0.087007

Hence (1 - d$^{(2)}$ /2)^2 = 0.912992

d$^{(2)}$ = 2*(1- 0.912992^(1/2))

= 8.8988%

(2)

(b) 1-d = (1-2*0.05) = 0.9

d = 0.1

Hence (1- d$^{(2)}$ /2)^4 = 0.9

d$^{(2)}$ =2*(1-0.9^(1/4))

= 5.199%

(2)

(iii) Let d be the simple annual discount rate. Then

(1-d*91/365) = (1+i)^(-91/365) where i =0.05

(1-d*91/365) = 0.98791

d*91/365 = 0.01206

d = 4.8493% **(2)**

[8]

Q3)

(i) It enables us to find the price of complex instrument instruments by ‘replicating’ the payoffs.

(2)

(ii) Present value of dividend is

60v^(1/2) at 5.0% + 60v at 5.5%

= 60*(1.05)^(-1/2) + 60* (1.055)^(-1)

= 60*(0.9759 + 0.9479)

= 115.428

Hence, forward price, F, is:

(550- 115.428)*1.055 = 458.4735

(4)

[6]

Q4)

(i) Total amount paid in 36 instalments = 550*36 = 19,800

The original purchase price = 15,000

Total interest paid over 3 year period = 19,800– 15,000 = 4,800

Flat Rate of interest = 4,800/(3*15,000)

= 10.67%

(2)

(ii) Let APR be the i such that

$12*550*a^{(12)}_{[3]} = 15{,}000$ at rate of interest i

$a^{(12)}_{[3]} = 15{,}000/6600$

$a^{(12)}_{3]} = 2.27273$

$a^{(12)}_{3]}$ at 20.0% = 2.29322

$a^{(12)}_{3]}$ at 21.0% = 2.26668

By interpolation

i = 20.77%

(3)

[5]

Q5)

(i)

- The bonds issued by governments with good rating are generally considered secure.
- Have low volatility of return relative to other investments
- Can provide guaranteed returns if held to maturity
- Can be risky in real term i.e. no protection against inflation

(2)

(ii)

(a) Let present value of the P.

P = 6*(1-0.30) * a(2)[20] + 120* v^20 - (120-P)*0.30* v^20 at 8%

a[20] = 9.8181, $i/i^{(2)}$ = 1.019615, v^20 = 0.21455

P = 4.2* 1.019615 * 9.8181 + 120* 0.21455 - (120-P)*0.30* 0.21455

P(1- 0.30*0.21455) = 42.04486 + 25.746*0.70 = 60.06706

P = 64.19924 **(4)**

(b) Since interest payments are at half yearly duration, it is easier to work with unit of half year. The effective rate per unit is 3%.

The duration = (? $t*C_t*v^t$)/ ? C_t*v^t)

The numerator for the bond is

$_1?^{40}\ 3*t*v^t + 40*120v^{40}$

$3(Ia)_{(40)}$ + 40*120*v40 ---------------------------(1)

]

$(Ia)_{(40)}$ = (adue(40) – 40v^40)/I

adue(40) = (1+i)*a(40) = 1.03*23.11477 = 23.80821

v^40 = 0.30656

$(Ia)_{(40)}$ = (23.80821 – 40*0.30656)/0.03 = 384.86033

(1) is 3*384.86033 + 40*120* 0.30656 = 2626.06899

The denominator for the bond is

$_1?^{40}\ 3*v^t + 120v^{40}$

= 3*a(40) +120v^{40}= 3* 23.11477 +120* 0.30656 = 106.13151

Hence, duration of bond is 2626.15899/106.13151 = 24.74 half-year

= 12.37 year

(4)

[10]

Q6)

(a) The factors are

Supply and demand

Base rate

Interest rates in other countries

Expected future inflation

Tax

Risk associated with changes in interest rate **(3)**

(b) The 10-year spot rate is measured by the yield on 10-year zero coupon bond. **(1)**

(c) The forward rate , $f_{t,r}$, is the annual rate of interest agreed at time 0 for an investment made at time t>0 for a period of r years. **(1)**

(d)

(i) We have:

$(1+y_3)^{3*}(1+f_3)$*$(1+f_{4,3})$^3 = $(1+y_7)$^7

$(1+f_3)$ = (1.055)^7/((1.065)^3* (1.057)^3)

f_3 = 1.45468/1.42651 –1 = 0.019747 =1.9747%

(ii) We have

$(1+y_4)^{4*}(1+f_{3,4})$^3 = $(1+y_7)$^7

$(1+y_4)$^4 = (1.055)^7/ (1.057)^3 = 1.23181

y_4 = 5.35%

(4)

[9]

Q7)

(i) Let X be the monthly instalment. Then,

$12*X*a^{(12)}[15]$ at 10%pa = 600,000

$a^{(12)}[15] = i/i^{(12)} * a[15]$, $i/i^{(12)} = 1.045045$, $a[12] = 7.6061$

$X = 600{,}000/(12*1.045045*7.6061)$
$= 6290.32$

(3)

(ii) After 1.5 years loan outstanding is

$12*6290.32*a^{(12)}[13.5] = 12*6290.32*1.045045*a(13.5)$
$= 12*6290.32*1.045045*7.2382$
$= 570{,}978.24$

Interest component of 19th monthly instalment = $i^{(12)}/12 * 570{,}978.24$
$= 0.095690/12*570{,}978.24 = 4553.08$

Capital component = $6290.32 - 4553.08 = 1737.24$

(4)

(iii)(a) Loan outstanding at the end of 5 years = $12*6290.32*a^{(12)}[10]$ at 10% pa interest
$= 12*6290.32*1.045045*6.1446 = 484{,}710.68$

The revised loan instalment Y is such that
$12*Y*a^{(12)}[10]$ at 8%pa = 484710.68
$a^{(12)}[10] = i/i^{(12)} * a[10]$, $i/i^{(12)} = 1.036157$ $a[10] = 6.7101$

$Y = 484{,}710.68 / (12*1.036157*6.7101) = 5809.61$

(4)

(iii)(b) Present value of reduction in monthly instalment is
$12*(6290.32 - 5809.61)*a^{(12)}[10]$ at 8%pa
$=12*480.71*1.036157*6.7101$
$= 40{,}106.89$

Processing fee = 4%* of original loan =0.04* 600,000
= 24,000
Thus, profit to borrower is 40,106.89 – 24,000= 16,106.89
And therefore the offer is profitable to the borrower.

(3)
[14]

Q8)

(i) **Call option**:
A call option gives you the right, but not the obligation, to buy a specified asset on set date in future for a specified price
Put option:
A put option gives to the right, but not the obligation, to sell a specifed asset on set date in future foe a specified price.

(2)

(ii) The difference is between ‘Right’ and ‘Obligation’. Selling a call means that you receive money and must sell the asset if holder of option wants to. Buying a put costs you money and gives you right to choose whether or not to sell the asset.

(2)

(iii) Let i be the real rate of interest per annum effective. Then

600= 30 [v^0.5/1.05^0.5 + v^1.5*1.08/1.05^1.5 + v^2.5*1.08^2/1.05^2.5 ---]
600= 30 * v0.5/1.05^0.5*[1+ v*1.08/1.05 + (v*1.08/1.05)^2 + ……]
600= 30 * v0.5/1.05^0.5 /(1- v*1.08/1.05)
600= 29.2770v^0.5 + 617.14286v …………………………eq(1)

The expected dividend yield is 60/600 = 5%. The dividend growth rate is 8%pa and inflation is 5%pa. Thus, the approximate real return on equity will be 8% (= dividend yield + growth rate – inflation)

So, at 8%, v = 0.925926
The RHS of eq(1) is 599.60

At 7.5%, v = 0.930233
The RHS of eq(1) is 602.3239

By interpolation
v = (600- 599.60)/(602.3239-599.60)*(0.930233-0.925926) +0.925926
v = 0.926558
i = 1/0.926558 –1 = 7.9263%

(5)
[9]

Q9)

(i) The discounted payback period of an investment project is the first time at which net present value of the cash flows from the project is positive.

(2)

(ii) Net present value
Net present value of the cost is :
PV(cost)= 10 + 1.5 [v + 1.05v^2 + 1.05^2*v^3 + 1.05^3v^4…….. 1.05^13v^14]
= 10 + 1.5/1.05*[(1.05v + (1.05v)^2 + (1.05v)^3……… (1.05v)^14]
= 10 + 1.5/1.05 [(v’) + (v’)^2 …… (v’)^14] where v’ = 1.05v
= 10+ 1.5/1.05 $a'_{[14]}$
where is evaluated at rate of interest i’ = 1.15/1.05-1 = 9.5238%.

Hence
$a'_{[14]}$ = (1- (1.095238^(-14))/0.095238 = 7.56188

PV(cost) = 10 + 7.56188*1.5/1.05 = 20.8026

Present value of income is:
PV(I) = 3.5*[v+v^2+v^3+ …….. v^15] + v5*0.50[v+2v^2+3v^3…..+10v^10]
= 3.5*a[15] + 0.50 * v^5 (Ia)[10] at 15%pa
= 3.5*5.8474 + 0.50* 1.15^(-5)* 21.9982 = 20.4658 +5.46851
= 25.93430

Hence, the present value of the project
=PV(I) – PV(cost) = 25.93430- 20.8026 = 5.1317 crore

(7)

(iii) Discounted Payback period

Present value of first 8 years cash flows
PV(cost) = 10 + 1.5[v + 1.05v^2 +1.05^2v^3 +…1.05^6v^7]
= 10 + 1.5/1.05a'[7]
= 10 + 1.5/1.05* [(1- (1.095238^(-7))/0.095238]
= 10 + 1.5/1.05*4.9457 = 17.06528

PV(I) = 3.5*a[8] + 0.50* v^5*(Ia)[3] @ 15%
= 3.5* 4.48732 + 0.50*1.15*(-5)*4.43440
= 15.70562 + 1.08245 = 16.78809
And
NPV = PV(I) – PV (cost) =16.78809 - 17.06528<0

Present value of first 9 years cash flows
PV(cost) = 10 + 1.5[v + 1.05v^2 +1.05^2v^3 +…1.05^7v^8]
= 10 + 1.5/1.05a'[8]
= 10 + 1.5/1.05* [(1- (1.095238^(-8))/0.095238]
= 10 + 1.5/1.05*5.42868 = 17.75526

PV(I) = 3.5*a[9] + 0.50* v^5*(Ia)[4] @ 15%
= 3.5* 4.77158 + 0.50*1.15*(-5)*6.641414
= 16.70053 + 1.650978 = 18.35152
And
NPV = PV(I) – PV (cost) =18.35152 – 17.75526 > 0

Therefore, discounted payback period is 9 years.

(4)
[13]

Q10)
(i) Let the accumulation of 1unit over 10 years period = S_{10}
$E(S_{10}) = E[(1+i_1)(1+i_2)(1+i_3)\ldots\ldots\quad(1+i_9)(1+i_{10})]$
Since i_t's are independent, this gives
$E(S_{10}) = E(1+i_1)\,E(1+i_2)\,E(1+i_3)\ldots\ldots\quad E(1+i_9)\,E(1+i_{10})$
= 1.08*1.08*1.08 ………… 1.08*1.08
= (1.08)^10 = 2.158925

$E(S_{10}^{\wedge 2}) = E[(1+i_1)\text{^}2\,(1+i_2)\text{^}2\,(1+i_3)\text{^}2\ldots\ldots\quad(1+i_9)\text{^}2\,(1+i_{10})\text{^}2]$

Since i_t's are independent, this gives
$E(S_{10}^{\wedge 2}) = E(1+i_1)\text{^}2\,E(1+i_2)\text{^}2\ldots\ldots\quad E(1+i_9)\text{^}2\,E(1+i_{10})\text{^}2$

$E(1+i_1)$^2= E(1+ 2i + i^2) = E(1)+2E(i) + E(i^2)

= 1+ 2E(i) + ?^2 +[E(i)]^2 = 1+2*.08 + 0.11^2 + 0.08^2

= 1.1785

$E(S_{10}{}^{\wedge 2})$ = 1.1785^10

$Var(S_{10}) = E(S_{10}{}^{\wedge 2})$ – $[E(S_{10})]$^2 = 1.1785^10 – 1.08^20

= 0.506725

$S.D(S_{10})$ = 0.711846

Thus, accumulation of 20,000 will have

Expected value of 20,000* 2.158925 = 43178.5

With standard deviation of (20000* 0.711846 = 14236.92

(9)

(ii) $(1+i_t)$? LN (?,?^{2})

$\ln(1+i_t)$? N (?,?^{2})

$\ln((1+i_t))$^10 = $\ln(1+i_t)$ $\ln(1+i_t)$ ……. $\ln(1+i_t)$? N (10?,10?^{2})

Thus,

$(1+i_t)$^10 ? LN (10?,10?^{2})

$E(1+i_t)$ = exp[?+?^{2}/2] = 1.08

$Var(1+i_t)$ = exp(2?+?^{2})[exp(?^{2})-1] = 0.11^2

1.08^2 *[exp(?^{2})-1] = 0.11^2

exp(?^{2}) = 0.11^2/1.08^2 +1 = 1.0103738

?^{2} = 0.01032036

exp[?+?^{2}/2] = 1.08

exp[?+0.0103203/2] = 1.08

? = ln1.08 – 0.01032036/2

? = 0.0718

S_{10} ? LN (10?,10?^{2})

$\ln S_{10}$? N (10?,10?^{2})

$\ln S_{10}$? N (0.718, 0.1032)

We require probability that S_{10} < 60%* 2.158925

i.e. S_{10} < 1.2954

Probability that $\ln S_{10}$ < ln(1.2954)

Probability that ($\ln S_{10}$ – 0.718) / ? 0.1032 < (ln(1.2954) – 0.718) / ? 0.1032

Pr(Z < - 1.42937) where Z ? N (0,1) = 0.0764

(8)

(iii) Let X be the sum invested such that

Pr (XS_{10} >=1.50*20000) =0.95

Pr (S_{10} >= 30000/X) = 0.95

Pr (ln S_{10} >= ln30000/X) = 0.95

Pr (Z >= (ln30000/X – 0.718)/ ? 0.1032) where Z ? N (0,1)= 0.95

Since we require 95% of populations to be more than value of z, we must have a negative z value

(In30000/X – 0.718)/ ? 0.1032 = -1.6449

ln30000/X = 0.189579

30000/X = 1.20874

X = 24819.23 **(4)**

[21]

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