# Institute of Actuaries of India

## Subject CT4 – Models

May 2012 Examinations

# INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

**Solution 1:**

**(a)** Commentary on extending various models to multiple states described

The Markov multiple state model is easily extended to cover the three states described in the sickness study. The estimators have the same simple form and statistical properties, depend only on data that will often be available exactly or approximately, and the apparatus needed in applications (such as the Kolmogorov equations) carries over without difficulty. Further extensions are possible, which complicate the calculation of probabilities but not the estimation of parameters, for example semi-Markov models.

The Poisson model extends just as easily to multiple decrements, but not to processes with increments. This model is an approximation to the multiple state model. If transition intensities are high, the Poisson model becomes a poorer approximation to the multiple-state model (because there is more randomness in the waiting times).

There are considerable difficulties in extending the binomial model even to multiple decrements. This is due to the fact that a binomial model is based on a series of independent Bernoulli trials, which have two possible outcomes. However, it may be possible to extend the life tables to multiple decrement tables, which could also be extended to incorporate increments. However, such extension of the binomial model itself is much more difficult.

**(b)** Commentary on other criteria that may be used to compare various models

Two other criteria that we might use to compare the three models are:

- how well each model represents the process we are trying to model; and
- how easy it is to find, characterize and use the model parameters given the data with which we must work.

**[5]**

**Solution 2:**

**a)** If the future lifetime of a life currently aged x is modeled as a random variable, the consistency condition states that the probability of surviving for time (s+t) after age x is given by multiplying:

- the probability of surviving for time s, and
- the probability of then surviving for a further time t

or by multiplying

- the probability of surviving for time t, and
- the probability of then surviving for a further time s.

i.e. the order in which we consider the two time periods between x, s and t is irrelevant.

Mathematically, this can be represented as follows:
s+tPx = tPx x sPt+x, where
tPx represents the probability of a life aged x surviving for a period t.

**b)** ${}_5p_0 = 0.14$
${}_{3.5}p_{1.5} = 0.77$
${}_{1.5}p_0 \times {}_{3.5}p_{1.5} = {}_5p_0$
${}_{1.5}p_0 = 0.1818$

Therefore, the probability of a patient surviving the first 18 months is 18.18%

**c)** The statement is incorrect.

The survival probability of 14% is for survival for 5 years after the surgery. This includes surviving the first 18 months, and then also surviving a further 3.5 years. Therefore, the survival probability in the first 18 months cannot be less than 14% and must indeed be greater than 14% (or equal to 14%, if there is a zero probability of death over the subsequent 3.5 years), since we are considering a longer period that the patient must survive, which includes the initial critical phase.

**[6]**

**Solution 3:**

**a)** The key steps in a modeling process can be described as follows:

- Develop a well-defined set of objectives that need to be met by the modeling process – *such as arriving at the premium rates for the term product at an acceptable degree of profitability*
- Plan the modeling process and how the model will be validated – *such as selecting the appropriate data, selecting suitable assumptions, building and reviewing the model etc*.
- Collect and analyze the necessary data for the model – *such as through company's internal experience, or industry data or census data or inputs from reinsurers*
- Define the parameters for the model and consider appropriate parameter values – *such as mortality, expenses, lapses etc*
- Define the model initially by capturing the essence of the real world system. Refining the level of detail in the model can come at a later stage – *such as setting out the required cash flow projections for this product*
- Develop the model – *build the cash flow projections*
- Test the reasonableness of the output from the model – *for instance, by comparing the premium rates for the product versus premium rates for other similar products of competitors*
- Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.
- Analyze the output from the model.
- Ensure that any relevant professional guidance has been complied with.
- Communicate and document the results and the model.

**b)** A good answer may cover following points (or any other suitable points):

- Sensitivity test allows one to appreciate the "What-If" situation. For Example, while modeling term product one needs to understand the criticality of mortality assumption and how sensitive the results are to changes in mortality assumption. If the results are very sensitive with respect to this assumption, then one needs to be careful and have sufficient confidence in deciding the assumption.
- It would help in identifying key risks (or most critical assumptions) – for example mortality assumption in respect of the term assurance product.
- It would help in generating a distribution of the key results (such as profitability measures) instead of just the base results.

**[8]**

**Solution 4:**

**a)** Let $Z_n$ = time between arrival of the $n^{th}$ and $(n-1)^{th}$ clients.

Then $Z_n$'s are i.i.d. exponential random variables with mean $\frac{1}{\lambda}$ i.e. $E[Z_n] = \frac{1}{\lambda}$

Let $T_n$ = arrival time of the $n^{th}$ client = $\sum_{i=1}^{n} Z_n$

$$E[T_n] = E[\sum_{i=1}^{n} Z_n] = \sum_{i=1}^{n} E[Z_n] = \frac{n}{\lambda}$$

The expected waiting time until the first client is allowed to see the tax consultant is

$E[T_3]=3/\frac{1}{10}$ =30 minutes

**b)** Let X(t) be the Poisson process with mean $\lambda$t. Note that $P(\text{ X}(t)) = \quad k = \frac{\lambda t^k}{k!} e^{\lambda t}$,

for
$\kappa = 1, \ldots \ldots \ldots \ldots \ldots$ We have

$P = P[\{$Nobody is allowed to see the consultant in the 1st hr$\}]$

$= P[\{$At most 2 clients in first 120 mins$\}]$

$= P[\{X(t) \leq over\ (0{,}120)\}]$

$= P[\ X(120) \leq 2]$

$= P[X(120) = 0\ ]+ P[X(120) = 1\ ]+ P[X(120) = 2\ ]$

$= e^{-\frac{120}{10}}+\left(\frac{120}{10}\right) e^{-\frac{120}{10}} + \frac{1}{2}\left(\frac{120}{10}\right)^2 e^{-\frac{120}{10}} \ = e^{-12}\ (1 + 12 + 72)$

$= 0.052\%$

**c)** In order to ensure that the consultant meets at least 8 clients in next three hours, 10 clients are required to visit the lawyer's office.

Hence, the Probability that the consultant would meet at least 8 clients is

$P = \quad P[\{$At least 10 clients arrive in 180 mins$\}]$

$= 1 - P[\{$At most 9 clients arrive in 180 mins $\}]$

$= 1 -\ P[\{X(t) \leq over\ (0{,}180)\}]$

$= 1 -\ P[\ X(180) \leq 9]$

$= 1 - \sum_{k=0}^{k=9} P[X(180) = i]$

$= 1 -\ P[X(180) = 0\ ]+ P[X(180) = 1\ ]+ P[X(180) = 2\ ]+\ldots\ldots+ P[X(180) = 9\ ]$

$= 1 - e^{-\frac{180}{10}}\left\{\sum_{0}^{9}\left(\frac{180}{10}\right)^k \times \frac{1}{k!}\right\}$

$$= 1 - e^{-\frac{180}{10}}(1 + 18 + \frac{18^2}{2!} + \cdots + \frac{18^9}{9!})$$

$$= 1 - 1.5\%$$

$$= 98.5\%$$

**[8]**

**Solution 5:**

**a)** A rate interval is defined as the period of one year during which a life's recorded age remains the same, e.g. the period during which an individual is "aged 36 last birthday" or "aged 42 nearest birthday".

The exact ages to which a rate interval applies if the age label is

- age nearest birthday is $[x - \frac{1}{2}, x + \frac{1}{2}]$; and
- age next birthday is $[x - 1, x]$

**b)** We can estimate the initial mortality rate by using the formula:

$\hat{q}_x = \dfrac{\theta_x}{E_x}$, where:

$\theta_x$ is the number of deaths for lives aged $x$ last birthday; and
$E_x$ is the initial exposed to risk for lives aged $x$ last birthday; and

**Number of deaths**

The valuation data is provided for in-force policies with age label "age next birthday". However, we can determine the in-force data for age last birthday by noting that if a life is aged $x$ next birthday, then it is aged $x - 1$ last birthday.

Given that annuities vest at exact age 70 only, and death is the only form of exit, we can deduce the number of deaths at each age from the in-force data:
For example, at 31.12.07, there were 45,780 policies in force for lives aged 71 next birthday (or aged 70 last birthday).

If there were no deaths in the year 2008 for these policies, we would expect 45,780 policies in force for lives aged 72 next birthday (or aged 71 last birthday) on 31.12.08. However, the actual number of policies is 44,819, thus implying 961 deaths [45,780 – 44,819] in the year.

Assuming births and deaths are spread uniformly through the year, the age labels that these deaths correspond to are:

- 480.5 deaths for age 70 last birthday; and
- **480.5 deaths for age 71 last birthday**
  in calendar year 2008.

Similarly, for age 72 next birthday (or 71 last birthday), there are 44,921 policies in force at 31.12.2007 and for age 73 next birthday (or 72 last birthday), there are 43,955 policies in force at 31.12.2008. This implies 966 deaths in 2008, split as:

- **483 deaths for age 71 last birthday; and**
- 483 deaths for age 72 last birthday
  in calendar year 2008.

Therefore, the total number of deaths for age 71 last birthday in calendar year 2008 is 480.5 + 483 = 963.5

We can thus calculate the total number of deaths in each calendar year for ages 71, 72 and 73 as follows:

**Number of deaths in each calendar year**

| | *Age last birthday* | | |
|---|---|---|---|
| | 71 | 72 | 73 |
| 2008 | 963.5 | 963.5 | 973.5 |
| 2009 | 968.5 | 965.5 | 974.5 |
| 2010 | 976.0 | 970.5 | 977.0 |
| 2011 | 988.5 | 978.0 | 981.5 |
| **Total** | **3896.5** | **3877.5** | **3906.5** |

**Exposed to risk**

We can use the census approximation to calculate the central exposed to risk as follows:

$$E_x^c \approx \begin{array}{l} \tfrac{1}{2}\left[P_x(31.12.07)+P_x(31.12.08)\right]+\tfrac{1}{2}\left[P_x(31.12.08)+P_x(31.12.09)\right] \\ +\tfrac{1}{2}\left[P_x(31.12.09)+P_x(31.12.10)\right]+\tfrac{1}{2}\left[P_x(31.12.10)+P_x(31.12.11)\right] \end{array}$$

Thus,

$E_{71}^c = 181315.0$; $E_{72}^c = 176318.0$ and $E_{73}^c = 171755.0$

Initial exposed to risk, $E_x \approx E_x^c + \dfrac{\theta_x}{2}$

Thus,

$E_{71} = 183263.25$; $E_{72} = 178256.75$ and $E_{73} = 173708.25$

**Initial mortality rate**

Finally, $\hat{q}_x = \dfrac{\theta_x}{E_x}$

Thus,

$$\hat{q}_{71} = \frac{3896.5}{183263.25} = 0.02126;$$

$$\hat{q}_{72} = \frac{3877.5}{178256.25} = 0.02175;$$

$$\hat{q}_{73} = \frac{3906.5}{173708.25} = 0.02249;$$

**[8]**

**Solution 6:**

**a)**

$$P_B{}'(t) = 0.9 * P_{\mathrm{A}}(t) + (-0.1) * P_{\mathrm{B}}(t)$$

$$P_B{}'(t) = 0.9 * P_{\mathrm{A}}(t) - 0.1 * P_{\mathrm{B}}(t)$$

**b)** As there are only two states,

$$P_{\mathrm{A}}(t) + P_{\mathrm{B}}(t) = 1$$

Substituting using the solution to (ii), we obtain

$$P_B{}'(t) = 0.9 * (1 - P_{\mathrm{B}}(t)) - 0.1 * P_{\mathrm{B}}(t)$$
$$P_B{}'(t) + P_{\mathrm{B}}(t) \ = 0.9$$

So that

$$\frac{d}{dt}\{e^t P_{\mathrm{B}}(t)\} = 0.9 * e^t$$

$$\{e^t P_{\mathrm{B}}(t)\} = 0.9 * e^t + C$$

Where $C = Constant$

Since $P_{\mathrm{B}}(0) = 0$ (boundary condition),

So,

$$P_{\mathrm{B}}(t) = 0.9(1 - \mathrm{e}^{-\mathrm{t}})$$

**c)** If $O_t$ is a random variable denoting the amount of time the book is available and $I_t$ is an indicator variable which takes the value 1 if available, 0 if borrowed then required expected value is

$E[O_t | P_{\mathrm{A}}(0) = 1]$

Since $P_{\mathrm{A}}(t) + P_{\mathrm{B}}(t) = 1$:

$P_{\mathrm{A}}(t) = 1 - \ 0.9(1 - \mathrm{e}^{-\mathrm{t}})$

$\therefore P_{\mathrm{A}}(t) = 0.1 + 0.9\mathrm{e}^{-\mathrm{t}}$

We have:

$E[O_t | P_{\mathrm{A}}(0) = 1] = \int_0^t E[I_s \, | P_{\mathrm{A}}(0) = 1] \mathrm{ds}$

$\qquad = \int_0^t P_{\mathrm{A}}(s) \, \mathrm{ds}$

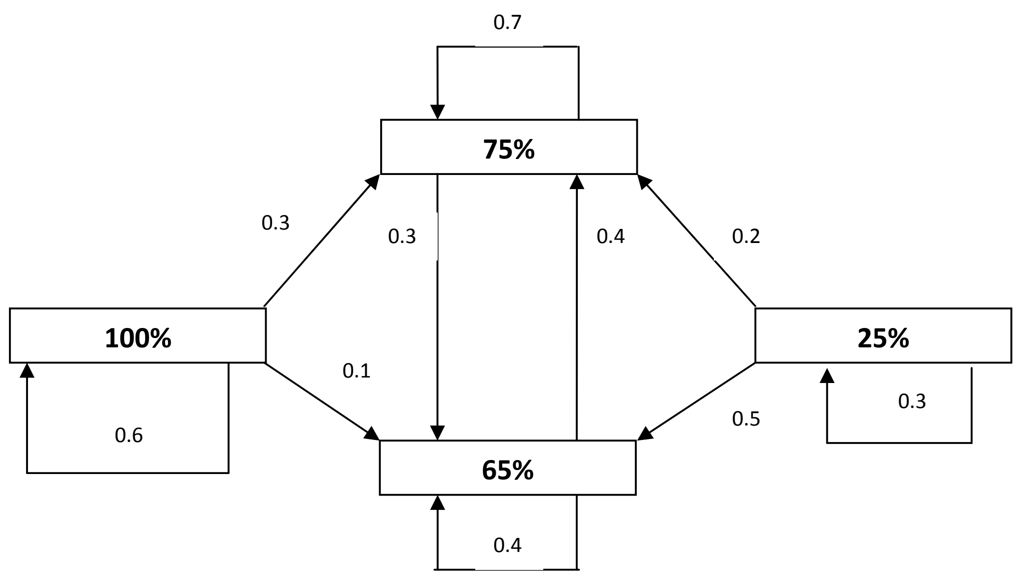
$$= \int_0^t (0.1 + 0.9 * \mathrm{e}^{-\mathrm{s}}) \, \mathrm{ds}$$
$$= (.1s - 0.9e^{-s})_0^t$$
$$= 0.1\mathrm{t} + 0.9 * (1 - \mathrm{e}^{-\mathrm{t}})$$

Since the transition rates are given on a fortnightly basis, the expected amount of time spent in one fortnight can be calculated by putting t = 1. This equals 0.6689 fortnight or 10.03 days.

**[9]**

**Solution 7:**

**a)** The state transition diagram is set out below:




**b)** We are to calculate for the % of corporate buyer having a target % for XYZ of 65%

- In 2 years time
- Over the long-run.

The state of the system after one year $S_1 = S_0 P$

$$(0.05,\ 0.30,\ 0.45,\ 0.20)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix} = (0.03,\ 0.445,\ 0.375,\ 0.15)$$

Hence the state of the system in 2 years time $S_2 = S_1 P$

$$(0.03, 0.445, 0.375, 0.15)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix} = (0.018,\ 0.5005,\ 0.3615,\ 0.12)$$

Hence the % of corporate buyer having a target % of 65% for XYZ in 2 years time is 36.15%.

The long run steady state can be found by solving the following equation:

$$S = SP$$

i.e.

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)P$$

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix}$$

Hence we have the five equations

$x_1 = 0.6x_1$

$x_2 = 0.3x_1 + 0.7x_2 + 0.4x_3 + 0.2x_4$

$x_3 = 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.5x_4$

$x_4 = 0.2x_3 + 0.3x_4$

$x_1 + x_2 + x_3 + x_4 = 1$

Now from the first equation above,

$$0.4x_1 = 0$$

So

$$x_1 = 0$$

Substituting this into the other equations above and rearranging we get

$0.3x_2 = 0.4x_3 + 0.2x_4 \quad (1)$

$0.6x_3 = 0.3x_2 + 0.5x_4 \quad (2)$

$0.7x_4 = 0.2x_3 \quad (3)$

$x_2 + x_3 + x_4 = 1 \quad (4)$

One of the above equations is redundant.

Hence we have

$$(x_1 = 0, x_2 = 0.5423, x_3 = 0.3559, x_4 = 0.1017)$$

**c)** The steady state proportion of customers having a 65% target allocation for XYZ Ltd is 35.59%.

It is noted that the initially this proportion is 45% and it quickly drops to:

- 37.5% in one year's time; and
- 36.2% in two years' time

It is, therefore, likely that the steady state shall be reached in a few years' time.

**[10]**

**Solution 8:**

**a)** A discrete-time Markov process with a finite state space is termed as Markov Chain.

The mathematical representation for a Markov chain is set out as follows:

$P(X_n = j | X_0 = i_0, X_1 = i_1 \dots X_{m-1} = i_{m-1}, X_m = i_m) = P(X_n = j | X_m = i_m)$

for all integer times n > m and states $i_0, i_1, \ldots i_{m-1}, i_m, j \text{ in } S$

The process $X_n$operates in discrete time space (hourly) and discrete state space (0, 1, 2 and 3). In addition, the future development of the process is independent of the past and therefore the process satisfies the Markov property and is therefore a Markov Chain.

**b)** Note that for $n = 2k$ which is even,

$$\{X_{2k} = 1 | X_0 = 1\} = \begin{pmatrix} \mathit{The\ transition}\ 1\ \mathit{to}\ 2\ \mathit{followed} \\ \text{by transition } 2\ \mathit{to}\ 1 \text{ and this is repeated} \\ k = \frac{n}{2}\ \ \mathit{times} \end{pmatrix}$$

So

$$p_{11}(n) = (p_{12}\, p_{21)}\, (p_{12}\, p_{21)} \dots \dots . . (p_{12}\, p_{21)}\, \mathit{for}\ k = \frac{n}{2}\ \ \mathit{times}$$

$$= \left\{\frac{1}{2} \times \frac{1}{2}\right\}^{\frac{n}{2}}$$

$$= \left\{\frac{1}{2}\right\}^{n}$$

Similarly,

$$p_{22}(n) = (p_{21}p_{12})(p_{21}p_{12}) \dots \dots . . (p_{21}p_{12})\mathit{for}\ k = \frac{n}{2}\ \ \mathit{times}$$

$$= \left\{\frac{1}{2}\right\}^{n}$$

Consider $p_{10}(n)$.

Where n=2:

$p_{10}(2) = p_{10}p_{00} = p_{10}$

Where n=4:

$p_{10}(4) = p_{10}p_{00}p_{00}p_{00} + p_{11}(2)p_{10}p_{00} = p_{10} + p_{11}(2)p_{10}$

Where n=6:

$p_{10}(6) = p_{10} + p_{11}(2)p_{10} + p_{11}(4)p_{10}$

Generalising:

$p_{10}(n) = p_{10}\{1 + p_{11}(2) + ..p_{11}(n-2)\}$

$\therefore p_{10}(n) = p_{10}\{1 + p_{11}(2) + ..p_{11}(2k-2)\}$

where 2k = n

$$\therefore p_{10}(n) = \frac{1}{2}\left\{1 + \left(\frac{1}{2}\right)^2 + ..\left(\frac{1}{2}\right)^{2k-2}\right\}$$

$$\therefore p_{10}(n) = \frac{1}{2}\left\{1 + \left(\frac{1}{4}\right)^1 + ..\left(\frac{1}{4}\right)^{k-1}\right\}$$

$$= \frac{1}{2} \times \frac{1 - \left(\frac{1}{4}\right)^k}{1 - \frac{1}{4}}$$

$$= \frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^k\right]$$

Similarly,

$p_{23}(n) = p_{23}\{1 + p_{22}(2) + ..p_{22}(n-2)\}$

$\therefore p_{23}(n) = p_{23}\{1 + p_{22}(2) + ..p_{22}(2k-2)\}$

$$\therefore p_{23}(n) = \frac{1}{2}\left\{1 + \left(\frac{1}{2}\right)^2 + ..\left(\frac{1}{2}\right)^{2k-2}\right\}$$

$$\therefore p_{23}(n) = \frac{1}{2}\left\{1 + \left(\frac{1}{4}\right)^1 + ..\left(\frac{1}{4}\right)^{k-1}\right\}$$

$$= \frac{1}{2} \times \frac{1 - \left(\frac{1}{4}\right)^k}{1 - \frac{1}{4}}$$

$$= \frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^k\right]$$

**c)** We calculate the remaining n-step transition probabilities $p_{ij}(n)$) row by row for n being even.

First row, $i = 0$:

$p_{00}(n) = 1$ as state 0 always changes to state 0

Since $\sum_{j=0}^{3} p_{0j}(n) = 1 \; so \; p_{01}(n) = p_{02}(n) = p_{03}(n) = 0$

Second row, $i = 1$

$p_{12}(n) = 0 \; as \; n \; is \; even$

$$p_{13}(n) = 1 - \sum_{j=0}^{2} p_{ij}(n)$$

$$= 1 - \left\{\frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right] + \left(\frac{1}{2}\right)^{2k} + 0\right\}$$
$$= \frac{1}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right]$$

Third row, $i = 2$,

$p_{21}(n) = 0 \; as \; n \; is \; even$

$$p_{20}(n) = 1 - \sum_{j=1}^{3} p_{2j}(n)$$

$$= 1 - \left[0 + \left(\frac{1}{2}\right)^{2k} + \frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right]\right]$$
$$= \frac{1}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right]$$

Fourth row, $i = 3$

$p_{33}(n) = 1$ as state 3 always changes to state 3

Since $\sum_{j=0}^{3} p_{3j}(n) = 1 \; so \; p_{30}(n) = p_{31}(n) = p_{32}(n) = 0$

In short, $for \; n = 2k$

$$P(n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right] & \left(\frac{1}{4}\right)^{k} & 0 & \frac{1}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right] \\ \frac{1}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right] & 0 & \left(\frac{1}{4}\right)^{k} & \frac{2}{3} \times \left[1 - \left(\frac{1}{4}\right)^{k}\right] \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**[14]**

**Solution 9:**

**a)** We can summarize the available data as follows:

| Life: S. no | Life: Lion name | Covariate, z: Sub-species | Age at exit | Reason for exit |
|---|---|---|---|---|
| 1 | Ajradanstra | 1 | 8 | Censored |
| 2 | Nala | 0 | 8 | Censored |
| 3 | Mufasa | 0 | 9 | Death |
| 4 | Simba | 0 | 9 | Censored |
| 5 | Bhasuraka | 1 | 10 | Death |
| 6 | Karalakesara | 1 | 11 | Censored |
| 7 | Scar | 0 | 12 | Death |
| 8 | Sarabi | 0 | 13 | Censored |

The partial likelihood, L(β) is given by:

$$L(\beta)=\prod_{j=1}^{k}\frac{\exp(\beta z_j)}{\sum_{i\in R(t_j)}\exp(\beta z_i)}$$

There are only 3 deaths in the data, at ages 9, 10 and 12. For age 9, there is a tie in the data, as there is one life censored at the same age. We assume that the censoring occurs just after the death is observed.

We can thus calculate the likelihood by determining the contribution to the partial likelihood from each death as follows:

$$L(\beta)$$

$$=\frac{\mu_3(9)e^{\beta.0}}{\mu_3(9)e^{\beta.0}+\mu_4(9)e^{\beta.0}+\mu_5(9)e^{\beta.1}+\mu_6(9)e^{\beta.1}+\mu_7(9)e^{\beta.0}+\mu_8(9)e^{\beta.0}}$$

$$\times\frac{\mu_5(10)e^{\beta.1}}{\mu_5(10)e^{\beta.1}+\mu_6(10)e^{\beta.1}+\mu_7(10)e^{\beta.0}+\mu_8(10)e^{\beta.0}}$$

$$\times\frac{\mu_7(12)e^{\beta.0}}{\mu_7(12)e^{\beta.0}+\mu_8(12)e^{\beta.0}}$$

$$=\frac{\mu_0}{\mu_0\left[1+1+e^{\beta}+e^{\beta}+1+1\right]}\times\frac{\mu_0e^{\beta}}{\mu_0\left[e^{\beta}+e^{\beta}+1+1\right]}\times\frac{\mu_0}{\mu_0[1+1]}$$

$$=\frac{1}{2e^{\beta}+4}\times\frac{e^{\beta}}{2e^{\beta}+2}\times\frac{1}{2}$$

$$\therefore L(\beta)=\frac{1}{8}\times\frac{1}{e^{\beta}+2}\times\frac{e^{\beta}}{e^{\beta}+1}$$

Taking logs and maximizing,

$$\ln[L(\beta)]=\ln\frac{1}{8}+\beta-\ln\left(e^{\beta}+2\right)-\ln\left(e^{\beta}+1\right)$$

$$\frac{\partial\ln[L(\beta)]}{\partial\beta}=1-\frac{e^{\beta}}{\left(e^{\beta}+2\right)}-\frac{e^{\beta}}{\left(e^{\beta}+1\right)}$$

Setting equal to zero,

$$\frac{\left(e^{\beta}+2\right)\left(e^{\beta}+1\right)-e^{\beta}\left(e^{\beta}+1\right)-e^{\beta}\left(e^{\beta}+2\right)}{\left(e^{\beta}+2\right)\left(e^{\beta}+1\right)}=0$$

$$\Rightarrow e^{2\beta}=2$$

$$\Rightarrow \beta=\tfrac{1}{2}\ln 2$$

i.e. $\beta$=0.34657

Checking for maxima,

$$\frac{\partial^2 \ln\left[L(\beta)\right]}{\partial\beta^2}=-\frac{e^{\beta}\left(e^{\beta}+2\right)-e^{2\beta}}{\left(e^{\beta}+2\right)^2}-\frac{e^{\beta}\left(e^{\beta}+1\right)-e^{2\beta}}{\left(e^{\beta}+1\right)^2}<0$$

So, the maximum likelihood estimate of $\beta$ is $\hat{\beta}=0.34657$

The hazard rates of African (z=0) and Asian (z=1) lions are in the same proportion at all times as a result of using the proportional hazard model.

Thus, we can determine the relative hazard rate between the two sub-species as:

$$\frac{\mu_0(x)e^{\beta.0}}{\mu_0(x)e^{\beta.1}}=\frac{1}{e^{0.34657}}=\frac{1}{1.414}$$

i.e. the hazard rates of Asian lions is approximately 1.4 times that of African lions.

**b)** We can estimate the variance of the maximum partial likelihood estimator $\tilde{\beta}$ using the approximation:

$$\operatorname{var}\left(\tilde{\beta}\right)\approx\left(-\frac{\partial^2 \ln L}{\partial\beta^2}\right)^{-1}\Bigg|_{\beta=\hat{\beta}}$$

$$\frac{\partial^2 \ln\left[L(\beta)\right]}{\partial\beta^2}=-\frac{2e^{\beta}}{\left(e^{\beta}+2\right)^2}-\frac{e^{\beta}}{\left(e^{\beta}+1\right)^2}$$

$$\therefore \operatorname{var}\left(\tilde{\beta}\right)=\left[\frac{2e^{\beta}}{\left(e^{\beta}+2\right)^2}+\frac{e^{\beta}}{\left(e^{\beta}+1\right)^2}\right]^{-1}\Bigg|_{\beta=0.34657}$$

$$\Rightarrow \operatorname{var}\left(\tilde{\beta}\right)=2.06066$$

As $\tilde{\beta}$ is asymptotically normally distributed, a 95% confidence interval for $\beta$ is:

$$\tilde{\beta}\pm 1.96\sqrt{\operatorname{var}\left(\tilde{\beta}\right)}=0.34657\pm\left[1.96\times\sqrt{2.06066}\right]=(-2.467, 3.160)$$

As this interval contains the value 0, we can conclude on the basis of the data provided that sub-species of lions (i.e. whether African or Asian) is not a significant covariate.

We can estimate the proportional hazard rates as follows:

African, wild [ $\underline{z}$ =0,0,0]:
$$\mu(x) = \mu_0(x).\exp(0.34657 \times 0 - 0.55 \times 0 - 0.40 \times 0) = \mu_0(x) \times 1$$

African, captive [ $\underline{z}$=0,1,0]: $\mu(x) = \mu_0(x).\exp(0.34657 \times 0 - 0.55 \times 1 - 0.40 \times 0) = \mu_0(x) \times 0.5769$

Asian, wild [ $\underline{z}$=1,0,0]: $\mu(x) = \mu_0(x).\exp(0.34657 \times 1 - 0.55 \times 0 - 0.40 \times 0) = \mu_0(x) \times 1.4142$

Asian, captive [ $\underline{z}$=1,1,1]: $\mu(x) = \mu_0(x).\exp(0.34657 \times 1 - 0.55 \times 1 - 0.40 \times 1) = \mu_0(x) \times 0.5469$

From this, we can conclude:

- In the wild, hazard rates of Asian lions is higher than that of African lions, by a factor of 1.4 [as before]
- However, in captivity the relative difference in the hazard rates reduces significantly – in fact, the hazard rates of Asian lions in captivity is less than that of African lions, by 5% [0.5469 / 0.5769]
- This also shows that the reduction in hazard rates in captivity is much more pronounced for Asian lions (hazard rates in captivity is 39% [0.5469 / 1.4142] of those in the wild) than that for African lions (for whom, the hazard rates in captivity is 58% compared to that in the wild).

**Solution 10:**

**a)** The graphical method of graduation is usually considered most appropriate under the following circumstances:

1. When a highly accurate answer is not essential

   Under such circumstances, the graphical graduation method may be used to provide a quick and reasonable answer. However, in this case, the insurance company intends to use the graduated rates for valuation as well as pricing purposes, for which it would be essential to have reliable and reasonably accurate rates. Therefore, the use of graphical graduation may not be appropriate given how the company intends to use the graduated rates.

2. When the data is scanty

   Graphical method may be a feasible way of carrying out graduation when the data available is not sufficient to graduate using the other methods. In the case of the insurance company, there seems to be sufficient data except for the younger ages from 20-29 years. There seems to be a large amount of exposure and deaths all other age groups for which the crude rates may be fairly reliable and the use of other methods of graduation possible.

   There is a mixed argument for using graphical graduation in the case of lower ages where there is little data available. The crude rate of 1.3514% for ages 20-24 is based on only one death so may not be very reliable. Therefore, using graphical method of graduation may provide the flexibility to use expert judgment on the most appropriate rates for these ages on the one hand, but given that the data is too scanty to draw any conclusions, this may expose the graduated rates to errors or bias on the part of the actuary performing the graduation.

Thus on the other hand, if using a parametric formula, we may be able to obtain a reliable closed form solution from higher ages and may be able to extrapolate where the data is scanty. Similarly, using standard tables may provide a reliable "reference rate" which may be appropriate since standard tables would be constructed using data from wider experience.

3. When special features need to be incorporated

   There may be some special features that need to be incorporated within the mortality rates – e.g. it is clear that the experience for younger ages is extra-ordinary. This could be either just be co-incidental or there may well be a genuine reason for this experience (e.g. some exclusion clauses for initial years of the policy resulting in lower claims). If this is the case, then graduation using graphical method provides the flexibility and possibility to allow for this within the graduation.

4. When it is not possible to use other methods due to some other reason

   If it is not possible to use any of the other methods, then graphical graduation might be the only option available. This could be the case, say, if it was not possible to calibrate the necessary parameters when using a formulaic approach (or too difficult / expensive to justify the effort). However, this seems unlikely as a glance through the crude rates suggests that these seem to have a visible trend (increasing with higher age bands, except for the outlier initially).

   Similarly, there could be a situation where a standard table is not available. Again, this seems unlikely as the company sells pure term assurance, mainly through banks, so there is a high probability that some standard table is available that can be suitably adjusted to allow for the company's experience.

   In summary, the case for using the graphical method seems quite weak for the insurance company given the purpose they would like to use the rates for and also based on a review of the available crude rates. Using a parametric formula or standard tables may be considered instead. However, using graphical graduation does offer the flexibility to use expert judgment where needed, so the company could even consider applying a parametric formula or standard table as the initial method and then "hand-polishing" the rates using a graphical method.

**b)** If the graduated rates are smooth but show little adherence to the data, then we say that the data may be over-graduated.

Similarly, if the graduated rates follow the crude rates closely but result in an irregular progression of over ages, we say that the data is under-graduated.

Therefore, to test the graduated rates for over/under-graduation, we need to test for both smoothness and adherence to data.

**Chi-square test**

We can test adherence to data using the chi square test.

The null hypothesis is:

H0: the graduated rates are the true underlying mortality rates for the population.

We can calculate the individual standardized deviations at each age using the approximation:

$$z_x = \frac{\theta_x - E_x\dot{q}_x}{\sqrt{E_x\dot{q}_x(1-\dot{q}_x)}} \approx \frac{\theta_x - E_x\dot{q}_x}{\sqrt{E_x\dot{q}_x}}$$

The approximation holds because $(1-\dot{q}_x) \approx 1$ for all $x$ since the $\dot{q}_x$ terms are small.

Thus,

| Age, $x$ | $E_x$ | $\theta_x$ | $\dot{q}_x$ | $z_x$ | $z_x^2$ |
|---|---|---|---|---|---|
| 20-24 | 120 | 1 | 0.1515% | 2.65 | 7.03 |
| 25-29 | 5982 | 12 | 0.2089% | -0.14 | 0.02 |
| 30-34 | 27839 | 65 | 0.2731% | -1.26 | 1.60 |
| 35-39 | 35487 | 124 | 0.3442% | 0.17 | 0.03 |
| 40-44 | 40859 | 156 | 0.4223% | -1.26 | 1.59 |
| 45-49 | 39850 | 220 | 0.5075% | 1.25 | 1.56 |
| 50-54 | 34859 | 189 | 0.6000% | -1.39 | 1.94 |
| 55-59 | 29349 | 210 | 0.7000% | 0.32 | 0.10 |
| | | | | **0.33** | **13.87** |

The test statistic for the chi-squared test is:

$\sum z_x^2$ = 13.87

Since the graduation was carried out graphically, we lose 2 or 3 degrees of freedom for every 10 age groups included in the graduation. We were given data from 8 age groups, so we are left with about 6 degrees of freedom.

From the tables, we find that the upper 5% point of $\chi_6^2$ is 12.59.

As the value of test statistic exceeds this, we reject the null hypothesis and conclude that the graduated rates do not provide a good fit to the data.

**Test for smoothness**

To test for smoothness, we can calculate the third differences of the graduated quantities.

The third differences can be calculated as follows:

| **Age,** $x$ | $\dot{q}_x$ | $\Delta\dot{q}_x = \dot{q}_{x+1} - \dot{q}_x$ | $\Delta^2\dot{q}_x = \Delta\dot{q}_{x+1} - \Delta\dot{q}_x$ | $\Delta^3\dot{q}_x = \Delta^2\dot{q}_{x+1} - \Delta^2\dot{q}_x$ |
|---|---|---|---|---|
| 20-24 | 0.1515% | 0.0574% | 0.0068% | 0.0001% |
| 25-29 | 0.2089% | 0.0642% | 0.0069% | 0.0001% |
| 30-34 | 0.2731% | 0.0711% | 0.0070% | 0.0001% |
| 35-39 | 0.3442% | 0.0781% | 0.0071% | 0.0002% |
| 40-44 | 0.4223% | 0.0852% | 0.0073% | 0.0002% |
| 45-49 | 0.5075% | 0.0925% | 0.0075% | |
| 50-54 | 0.6000% | 0.1000% | | |
| 55-59 | 0.7000% | | | |

The criterion of smoothness usually used is that the third differences of the graduated rates should:

a. be small in magnitude compared with the quantities themselves; and
b. progress regularly.

For this graduation, both these conditions are met, which indicates that the graduated rates are very smooth.

**Conclusion**

From the above two tests, we can see that the graduated rates:

- do not meet the chi square test for adherence to data; and
- are smooth.

Thus, the graduation seems to have led to the data being over-graduated.

**c)** We concluded above that the rates were over-graduated based on the observation that the chi-square test indicated that the graduated rates do not adhere to the data. However, a closer look at the standardized deviations indicates that more than half the value of test statistic came from the first age group of 20-24 years [7.03 out of a test statistic of 13.87]. The data in respect of this age group is particularly scanty (only one death in the last ten years!); therefore the crude rates could well be unreliable. Ignoring this outlier, the graduated rates in fact show a very good adherence to data, as the remaining standardized deviations are quite small. In this case, the graduation seems to be neither over nor under-graduated but may be considered adequate; as the graduation is both smooth as well as adheres well to the crude data (except for the dodgy first age group data).

Using the chi-square test to determine the adherence to data in case of graphical graduation has a limitation because it is necessary to determine the number of degrees of freedom for the test statistic. The number of degrees of freedom to use when graphical graduation has been used is not obvious.

The graduating curve has to a certain extent been forced to fit the rough data but it is subjective as to how many degrees of freedom should be deducted for this.

The chi square test is thus approximate and any result should be considered intelligently and not just blindly accepted.

**[19]**

**[TOTAL MARKS – 100]**

*******************************