Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2012 Examinations

INDICATIVE SOLUTIONS

Soln 1)

a)(i)

$$i=\left(1+\frac{i^{(4)}}{4}\right)^4-1=(1+.02)^4-1=8.2432\%$$

a)(ii)

$$d^{(12)}=12\left(1-\left(\frac{1}{1+i}\right)^{(1/12)}\right)=7.8949\%$$

$$\ddot{s}_{\overline{5}|}^{(12)}=\frac{(1+i)^5-1}{d^{(12)}}$$

$$=\frac{(1+8.2432\%)^5-1}{7.8949}$$

=6.1552

b)

$$(I\bar{a})_{\overline{n}|}=\int_0^n \lceil t\rceil v^t\,dt$$

$$=\int_0^1 v^t\,dt+\int_1^2 2v^t\,dt+......+\int_{n-1}^n nv^t\,dt$$

$$=\bar{a}_{\overline{1}|}+2v\bar{a}_{\overline{1}|}+.....+nv^{n-1}\bar{a}_{\overline{1}|}$$

Since $\int_{n-1}^n v^t\,dt=\left[\frac{v^t}{\ln(v)}\right]_{n-1}^n=\left[\frac{v^t}{\ln(1+i)^{-1}}\right]_{n-1}^n=\left[\frac{v^t}{(-\delta)}\right]_{n-1}^n=\frac{v^{n-1}-v^n}{\delta}=v^{n-1}\frac{1-v}{\delta}=v^{n-1}\bar{a}_{\overline{1}|}$

(Full credit should be given even if this explanation is not given by the student)

$$=\bar{a}_{\overline{1}|}(1+2v+.....+nv^{n-1})$$

$$=\frac{\bar{a}_{\overline{1}|}}{v}(v+2v^2+.....+nv^n)$$

$$=\frac{\bar{a}_{\overline{1}|}}{a_{\overline{1}|}}(v+2v^2+.....+nv^n),\ \text{since}\ a_{\overline{1}|}=v$$

$$= (\mathrm{I}a)_{\overline{n}|} * \left(\frac{\bar{a}_{\overline{1}|}}{a_{\overline{1}|}} \right)$$

Alternate solution

$$(\mathrm{I}\bar{a})_{\overline{n}|} = \int_0^1 \exp(-\delta t)\,dt + \int_1^2 2\exp(-\delta t)\,dt + \ldots + \int_{n-1}^{n} n\exp(-\delta t)\,dt$$

$$= \frac{1}{\delta}\left\{\left[-v^t\right]_0^1 + \left[-2v^t\right]_1^2 + \ldots + \left[-nv^t\right]_{n-1}^{n}\right\}$$

$$= \frac{1}{\delta}\left\{(1-v) + (2v - 2v^2) + \ldots. + (nv^{n-1} - nv^n)\right\}$$

$$= \frac{1}{\delta}\left\{(1 + 2v + \ldots. + nv^{n-1}) - (v + 2v^2 + \ldots. + nv^n)\right\}$$

$$= \frac{1}{\delta} * \left(\frac{1}{v} - 1\right) * (v + 2v^2 + \ldots. + nv^n)$$

$$= \frac{i}{\delta}(\mathrm{I}a)_{\overline{n}|}$$

$$= (\mathrm{I}a)_{\overline{n}|} * \frac{1-v}{\delta} * \frac{i}{1-v}$$

$$= (\mathrm{I}a)_{\overline{n}|} * \left(\frac{\bar{a}_{\overline{1}|}}{a_{\overline{1}|}} \right)$$

[8]

Sol. 2)

Present value calculation for the continuous payment

$$500 * \int_2^4 e^{(0.1t + 0.001t^2)} * e^{-\int_0^t (0.07 + 0.002s)\,ds}\,dt$$

$$= 500 \int_2^4 e^{(0.1t + 0.001t^2)} * e^{-\left[0.07s + 0.001s^2\right]_0^t}\,dt$$

$$= 500 \int_2^4 e^{(0.1t+0.001t^2)} * e^{-0.07t-0.001t^2} dt$$

$$= 500 \int_2^4 e^{0.03t} dt$$

$$= \frac{500}{0.03} \left[e^{0.03t}\right]_2^4$$

$$= \frac{500}{0.03} \left(e^{0.12} - e^{0.06}\right)$$

$$= 1094.3384$$

Present value calculation for the fixed payment of Rs.15,000/- at time $t = 6$

$$15,000 * e^{-\int_0^5 (0.07+0.002s)ds} * e^{-\int_5^6 (0.085-0.001s)ds}$$

$$= 15,000 * e^{-\left[0.07s+0.001s^2\right]_0^5} * e^{-\left[0.085s-0.0005s^2\right]_5^6}$$

$$= 15,000 * e^{-(0.35+0.001*25)} * e^{-(0.085*6-0.0005*36-0.085*5+0.0005*25)}$$

$$= 15,000 * e^{-0.375} * e^{-0.0795}$$

$$= 9521.4791$$

The price the investor should pay

$$= 1094.3384 + 9521.4791 = ₹10,615.8175 \text{ /-}$$

[7]

Sol. 3)

a) Main differences between preference share and ordinary shares are

- A preference share pays a dividend which is limited to a set amount which is almost always paid whereas an ordinary share pays a dividend out of residual profits, if any, and the dividend amount for ordinary share has no minimum or maximum values.
- Preference shares ranks above ordinary shareholders, both on dividends and capital repayment on winding up.
- Dividends under both preference shares and ordinary shares are paid at the directors'

discretion, but no ordinary dividend can be paid if there are any outstanding preference dividends.

- Ordinary shares have voting rights in proportion to number of shares held but preference shareholder gets voting rights if dividends are unpaid.
- Expected return likely to be lower than on ordinary shares due to lower risk.
- Less common than ordinary shares and hence, marketability lower than that of ordinary shares.

b)

A currency swap is an agreement to exchange a fixed series of interest payments and a capital sum in one currency for a fixed series of interest payments and a capital sum in another. The initial exchange of principal would usually be based on current spot rates.

[6]

Sol. 4)

Value of the forward contract is given by

$$f = S_r - I - Ke^{-\delta(T-r)}$$

Where,

r = Present time

T = Time to maturity of the forward contract

Thus, (T-r) = 1

δ =Continuously compounded risk-free rate of interest for the interval from r to T

= ln(1+5.5%) = 5.35%

S_r = Spot price of the security at time r = 105

I = Present value, at the risk-free interest rate, of the income generated by the security during the interval from r to T

K = Delivery price of the forward contract = 112

f = The value of a long position in the forward contract

Working in per ₹ 100 nominal

$$I = 3(1+5\%)^{-0.5} + 3(1+5.5\%)^{-1} = 2.9277 + 2.8436 = 5.7713$$

$$f = 105 - 5.7713 - 112(1+5.5\%)^{-1} = 99.2287 - 106.1611 = -6.9324$$

The value of the investor's short position in the forward contract on ₹ 10 lacs is therefore

$$\left(\frac{10,00,000}{100}\right)*\left(-f\right)=69,324$$

[5]

Sol.5)
a) Interest rates vary over time mainly due to following factors which are not normally constant over time:

- Supply and demand
- Base rates
- Interest rates in other countries
- Expected future inflation
- Tax rates

b)(i)

More issue of fixed interest bonds => supply increases => prices fall => yields rise

b)(ii)

Invest more in fixed interest bonds => demand rises => prices rise => yields fall

[4]

Sol. 6)

Cash-flows to the Insurance Company:

A positive cash-flow at start of the plan;

Followed by a series of negative cash-flows at the start of each year in future, over the next 25 years.

The number of future negative cash-flows to the Insurance Company is known at the start of the plan.

The amount of future negative cash-flows to the Insurance Company will depend on the inflation. Hence they are not known at the start of the plan.

[3]

Sol. 7)

a)

(i) $E(i) = 5\% * 0.2 + 7\% * 0.45 + 9\% * 0.35 = 7.30\%$

Thus, amount invested = $20{,}000 * (1 + 7.30\%)^{-5} = 14{,}061.49$

(ii) Expected profit=

$$14{,}061.49 * \left((1+5\%)^5 * 0.2 + (1+7\%)^5 * 0.45 + (1+9\%)^5 * 0.35\right) - 20{,}000$$

$$= 14{,}061.49 * \left(1.27628 * 0.2 + 1.40255 * 0.45 + 1.53862 * 0.35\right) - 20{,}000$$

$$= 14{,}061.49 * \left(0.25526 + 0.63115 + 0.53852\right) - 20{,}000$$

$$= 3{,}589.34 + 8{,}874.91 + 7{,}572.39 - 20{,}000$$

$$= 20{,}036.64 - 20{,}000$$

$$= 36.64$$

b)

Return i has mean of 0.017 and variance 0.0004

Thus , (1+i) is log –normally distributed with mean 1.017 and variance 0.0004

$1.017 = e^{\mu + \sigma^2/2}$

$0.0004 = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$

$\frac{0.0004}{1.017^2} = e^{\sigma^2}$ -1

$\sigma^2 = \log_e$ (0.0003867 + 1) = 0.0003867

$\mu = \log_e$ (1.017) - $\frac{0.0003867}{2}$ =0.016664

Thus, $\log_e(1+i) \sim N(1.6664\%, 1.97\%^2)$

From the tables, we know that the value of standard normal variate, such that the probability is greater than 80% is -0.84. Thus $Z = \left(\frac{\log_e(1+i) - 1.6664\%}{1.97\%}\right) \sim N(0,1)$

$$P\left(\frac{\log_e(1+i)-1.6664\%}{1.97\%} > -0.84\right) = 0.80$$

$$P(\log_e(1+i) > 0.000116) = 0.80$$

$$P(i > 0.00012) = 0.80$$

thus, k = 0.012% or approx 0%

[9]

Sol. 8)

Lump sum payable to the purchaser on 1st April,2012 for each unit of financial instrument

= 1,500*(1+10.50%)^2 *(1+9.50%)^2+1500*(1+10.50%)^2 * (1+9.50%) +1,500*(1+10.50%)^2

+ 1500 *(1+10.50%)

=2,196.06 + 2,005.53 + 1,831.54 + 1,657.50

= ₹ 7,690.63/-

The purchaser paid ₹ 1,500/- per annum at the beginning of each year in rupee and subsequently the amount was invested in $. Also At the end of 4 years the accumulated invested amount has to be calculated in rupee.

Following table shows accumulated values (in $) of the premiums on 01/04/2012

Date	Amount of premium in ₹	Exchange rate (₹ per $)	Amount of premium in $	Crude oil price in $ (per barrel)	Amount of barrel bought
01/04/2008	1,500	45.5	32.967	55	0.5994
01/04/2009	1,500	48.1	31.185	53.5	0.5829
01/04/2010	1,500	50	30	70	0.4286
01/03/2011	1,500	48	31.25	90	0.3472
01/04/2012	-	49.5	0	111	0

Total barrel bought per unit of instrument = 1.9581

Total value in $ as on 01/04/2012 = 1.9581*111 = 217.3491

Total value in rupee as on 01/04/2012 = 217.3491 * 49.5 = 10,758.78

Profit per unit bond= ₹ 10,758.78 – ₹ 7,690.63 = ₹ 3,068.15/-

[7]

Sol. 9)

a)

Capital Expenditure at start = ₹ 50 Lacs

Per policy yearly cost = ₹ 500/-, incurred at start of the year

Commission as % of premium = 20%, paid at start of the year

Average premium per policy = ₹ 5,000/-, earned at start of the year

Average claim size = 5 * 5,000 = ₹ 25,000/-, incurred at the year end

Claim probability = 5%

Effective rate of interest = 7.50% ; Hence $v = (1 + i)^{-1} = 0.930233$

Therefore equation of value till the end of seventh year

$-50*10^5+v^2(x+2x*v+3x*v^2+4x*v^3+5x*v^4)*(-500+5{,}000*(1-20\%)-5\%*25{,}000*v) = 0$

Where x is the number of policies to be sold in the third year.

Hence,

$x * v^2(1+2v+3v^2+4v^3+5v^4)*(-500+5{,}000*(1-20\%)-5\%*25{,}000*v) = 50*10^5$

$v^2(1+2v+3v^2+4v^3+5v^4) = 10.747721$

$-500+5{,}000*(1-20\%)-5\%*25{,}000*v = 2{,}337.2088$

b)

$x = 50*10^5/(10.747721 * 2{,}337.2088) = 199.05 \approx 200$ minimum number of policies

For viability of the project following condition needs to be satisfied

$-50*10^5+200 *v^2(1+2*v+3*v^2+4*v^3+5* v^4* \ddot{a}_{\overline{\infty}|})*(-500+5{,}000*(1-20\%)-5\%*25{,}000*v) \geq 0$

Where i = 18% per annum and $v = (1+i)^{-1}=(1+18\%)^{-1}=0.847458$

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d} = \frac{1}{1-v} = 6.555556$$

Hence LHS

$= -50*10^5+200 *v^2(1+2*v+3*v^2+4*v^3+5*v^4* 6.555556)*(-500+5{,}000*(1-20\%)-5\%*25{,}000*v)$

$v^2(1+2*v+3*v^2+4*v^3+5*v^4* 6.555556) = 17.373214$

$(-500+5{,}000*(1-20\%)-5\%*25{,}000*v) = 2{,}440.6775$

Hence LHS = -50*10^5+200 *17.373214 *2,440.6775 = ₹34,80,483/- > 0

Hence the project is viable.

[10]

Sol. 10)

a)

Present value of the liabilities (PV_L)

$$20,00,000 * \bar{a}_{\overline{10}|} = 2,000,000 * \frac{1-v^{10}}{\delta}$$

$$= 2,000,000 * \frac{1-(1+7\%)^{-10}}{\ln(1+7\%)}$$

=2,000,000 * 7.2666

=1,45,33,200/-

Duration of the liabilities is given by

$$\frac{20,00,000 * \int_0^{10} t v^t dt}{20,00,000 * \int_0^{10} v^t dt} = \frac{\left(\bar{I}\bar{a}_{\overline{10}|}\right)}{\bar{a}_{\overline{10}|}}$$

$$= \frac{1}{\bar{a}_{\overline{10}|}} * \frac{\bar{a}_{\overline{10}|} - 10v^{10}}{\delta}$$

$$= \frac{1}{7.2666} * \frac{7.2666 - 10*(1+7\%)^{-10}}{\ln(1+7\%)} = \frac{1}{7.2666} * 32.2665$$

=4.4404 years

b)

Let the nominal amount of zero-coupon bond invested be A

Let the nominal amount of 7-year bond invested be B

Thus, the present value of the two bonds (PV_A) will be

$$A * v^3 + B*(8\% * a_{\overline{7}|} + v^7),$$

Where $a_{\overline{7}|} = \frac{1-v^7}{i} = \frac{1-(1+7\%)^{-7}}{7\%}$ =5.389289

= $A*(1+7\%)^{-3} + B*\left(8\%*5.389289 + (1+7\%)^{-7}\right)$

= 0.816298 A + 1.053893 B

As the above present value of assets is equal to the present value of the liabilities, we have

0.816298 A + 1.053893 B = 1,45,33,200 ---------(1)

Similarly the duration of the assets comprising the two bonds will be

$\frac{1}{\mathrm{PV_A}}\left(3*A*v^3 + B*\left(8\%*\left(Ia_{\overline{7}|}\right)+7v^7\right)\right)$, where $i =$ 7%

Where $Ia_{\overline{7}|} = \frac{\ddot{a}_{\overline{7}|} - 7v^7}{i}$ = 20.104154

since $\ddot{a}_{\overline{7}|} = (1+i)a_{\overline{7}|} = (1+i)*5.389289 =$ 5.766539

= $\frac{1}{\mathrm{PV_A}}\left(2.448894*A + B*\left(8\%*20.104154 + 7*(1+7\%)^{-7}\right)\right)$

= $\frac{1}{\mathrm{PV_A}}\left(2.448894*A + B*\left(1.608332 + 4.359248\right)\right)$

= $\frac{1}{\mathrm{PV_A}}\left(2.448894*A + 5.96758*B\right)$

As the above duration of assets is also equal to that of the liabilities, we have

$\frac{1}{\mathrm{PV_A}}\left(2.448894*A + 5.96758*B\right) = 4.4404$

$\Rightarrow \left(2.448894*A + 5.96758*B\right) = 4.4404*\mathrm{PV_A} = 4.4404*1,45,33,200$

$\Rightarrow 2.448894*A + 5.96758*B = 6,45,33,221$ --------(2)

Multiplying (1) with 3 and subtracting from (2)

2.805901 B = 2,09,33,621

B = 74,60,570

Replacing value of B in (1)

A = (1,45,33,200 - 1.053893 * 74,60,570)/0.816298

= 81,71,719

Hence, using equation 1, amount to be invested in zero coupon bond

= 81,71,719* 0.816298 = 66,70,558

Similarly, using equation 1, Amount invested in coupon paying bond

= 74,60,570* 1.053893 = 78,62,642

[14]

Sol. 11)

a)

Coupon after tax, (1-t)g = (1 – 20%) * 6% = 4.8%

Minimum yield required convertible half-yearly, $i^{(2)} = 2*(1+4.8\%)^{0.5} - 1) = 4.7437\%$

Since $i^{(2)} < (1-t)g$, the worst case will be to redeem earliest. Hence price should be calculated assuming earliest redemption date i.e. on 1st July 2014.

Total number of coupon payments between 1st September 2006 and 1st July 2014 = 16

Price = 10,000 $(1+i)^{2/12}\left\{(1-t)g * a_{\overline{8}|}^{(2)} + v^8)\right\}$, where i = 4.8%

$$a_{\overline{8}|}^{(2)} = \frac{1-(1+i)^{-8}}{i^{(2)}} = \frac{1-(1+4.8\%)^{-8}}{4.7437\%} = 6.593122$$

Hence, Price = 10,000 * $(1+4.8\%)^{(2/12)}$ *((1-20%)*6% * 6.593122 + $(1+4.8\%)^{-8}$)

= 10,115.86

b)(i)

Coupon after tax, (1-t)g = (1 – 25%) * 6% = 4.5%

Minimum yield required convertible half-yearly, $i^{(2)} = 2*(1+5\%)^{0.5} - 1) = 4.939\%$

Since $i^{(2)} > (1-t)g$, the worst case will be to redeem latest. Hence price should be calculated assuming latest redemption date i.e. on 1st July 2018. Also there is capital gain to second investor.

Total number of coupon payments between 1st July 2012 and 1st July 2018 = 12

Price, P = $10{,}000\left\{(1-t)g * a_{\overline{6}|}^{(2)} + v^6\right\} - 30\% * (10{,}000 - P)v^6$, where i = 5%

$$\Rightarrow P(1-30\%v^6) = 10{,}000\left\{(1-t)g * a_{\overline{6}|}^{(2)} + v^6\right\} - 30\% * 10{,}000v^6$$

$$a_{\overline{6}|}^{(2)} = \frac{1-(1+i)^{-6}}{i^{(2)}} = \frac{1-(1+5\%)^{-6}}{4.939\%} = 5.13838$$

Hence,

$P(1-30\%*(1+5\%)^{-6}) = 10{,}000\ \{(1-25\%)*6\% * 5.13838 + (1+5\%)^{-6}\} - 30\% * 10{,}000 * (1+5\%)^{-6}$

=>P * 0.7761 = 7,535.78

=> P = 9,709.81

b)(ii)

Total number of coupon payments between 1st September 2006 and 1st July 2012 = 12

Assuming that the annual effective rate of return to first investor as i the equation of value is

$$10{,}115.86 = (1+i)^{(2/12)}(10{,}000*(1-20\%)6\% * a_{\overline{6}|}^{(2)} + 9{,}709.81*(1+i)^{-6})$$

$$\Rightarrow 10{,}115.86 = (1+i)^{(2/12)}(480 * a_{\overline{6}|}^{(2)} + 9{,}709.81*(1+i)^{-6})$$

Now if i = 4% , $i^{(2)}$ = 3.9608% , $a_{\overline{6}|}^{(2)}$ = 5.294 , and RHS = 10,281.9154

if i = 5% , $i^{(2)}$ = 4.9390% , $a_{\overline{6}|}^{(2)}$ = 5.1384 , and RHS = 9,791.3390

Hence by interpolation

$$\frac{i-4\%}{5\%-4\%} = \frac{10{,}115.86 - 10{,}281.9154}{9{,}791.3390 - 10{,}281.9154} = 0.3385$$

i = 4% + 0.3385 * (5% - 4%) = 4.339%

Hence, the annual effective rate of return earned by the first investor during the period is 4.3% (correct up to one decimal place).

[13]

Sol.12) a) Interest rate charged by the bank, $i^{(4)}$=7.5% per annum

Hence effective rate of interest per annum,

$i = (1+i^{(4)}/4)^4 - 1 = 7.7136\%$

Calculation for Option (i)

Loan amount after 2 years i.e. at the time of first installment payment

$L_2 = L_0(1+i)^2$= 10 Lacs * (1 + 7.7136%)^2 = 11.6022 Lacs

The first installment amount X is given by the equation

$2X\,\ddot{a}^{(2)}_{\overline{10}|} = L_2$

$$d^{(2)} = 2\left(1-\left(1-\frac{i}{1+i}\right)^{(1/2)}\right) = 7.2942\%$$

$$\ddot{a}^{(2)}_{\overline{10}|} = \frac{(1-(1+i)^{-10})}{d^{(2)}} = 7.1885$$

$$X = \frac{L_2}{2*\ddot{a}^{(2)}_{\overline{10}|}}$$ = 11.6022 Lacs/(2*7.188)= 0.807 Lacs =₹ 80,700/-

Calculation for Option (ii)

Loan amount after 2 years i.e. at the time of first installment payment

$L_2 = L_0$ = 10 Lacs, as the interest on loan amount have been paid over the 2 years

The first installment amount X is given by the equation

$(X - 10{,}000)\ \ddot{a}_{\overline{10}|} + 10{,}000\ I\ddot{a}_{\overline{10}|} = L_2$

$$\ddot{a}_{\overline{10}|} = (1+i)\frac{(1-(1+i)^{-10})}{i} = 7.322$$

b)

$$I\ddot{a}_{\overline{10}|} = (1+i)\frac{\ddot{a}_{\overline{10}|} - 10(1+i)^{-10}}{i} = 35.8239$$

$$X = \frac{L_2 - 10{,}000 I\ddot{a}_{\overline{10|}} + 10{,}000\ddot{a}_{\overline{10|}}}{\ddot{a}_{\overline{10|}}}$$ = 0.97648 Lacs = ₹ 97,648/-

In case of Option (ii) Akash need to pay

$2*i*L_2+10*X+10{,}000*(1+2+3+\ldots+9) = 2iL_2 + 10\,X + 10{,}000 * (0.5 * 9 * (9+1))$

c)

=2*7.7136% * 10 Lacs + 10*0. 97648 Lacs + 10,000 * 9 * (10/2)

=15.80752 Lacs

=₹ 15,80,752/-

In case of Option (ii)

Amount payable at 9^{th} installment = 0. 97648 Lacs + 8 * 10,000 = 1.77648 Lacs

Amount payable at 10^{th} installment =0. 97648 Lacs + 9 * 10,000 = 1.87648 Lacs

Hence,

Loan capital outstanding just after payment of 8^{th} installment

= $(1.87648 \text{ Lacs}/(1+7.7136\%)^2) + (1.77648 \text{ Lacs}/(1+7.7136\%))$ = 3.26661 Lacs

Interest element of 9^{th} installment payment

= 3.26661 Lacs * 7.7136% = 0.25198 Lacs

Hence capital element of 9^{th} installment payment

= 1.77648 Lacs - 0.25198 Lacs = 1.5245 Lacs = ₹ 1,52,450/-

[14]

[TOTAL MARKS – 100]

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