INSTITUTE OF ACTUARIES OF INDIA

CT8 – Financial Economics

MAY 2011 EXAMINATIONS

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Q1.) **(i)** F(t,T,S) is forward rate at t for delivery between T and S.

In an arbitrage-free market, it represents the force of interest at which investors can agree at time *t* to borrow or lend over the period from *T* to *S* .

(ii) The value of a bond in general is given by:

Value of bond = PV of coupons + PV of maturity proceeds

We know rating specific forward rates and we need to calculate the value of the bonds one year from now. Therefore

$$\text{Value of bond} =$$

$$\text{Coupon} * \left\{1 + e^{-(2-1)*F(0,1,2)} + e^{-(3-1)*F(0,1,3)} + e^{-(4-1)*F(0,1,4)}\right\} + \text{ Maturity Proceeds} * e^{-(4-1)*F(0,1,4)}$$

where the forward rates correspond to the rating at the end of the year.

The value of the A3 rated bond at the end of the first year if its rating remains unchanged is:

$$= 10^6\{0.1 * \{1 + 0.902578 + 0.786628 + 0.666977\} + 1.0 * 0.666977\}$$

$$= 1{,}002{,}595$$

The value of the B2 rated bond at the end of the first year if its rating remains unchanged is:

$$= 10^6\{0.15 * \{1 + 0.886920 + 0.755784 + 0.582748\} + 1.0 * 0.582748\}$$

$$= 1{,}066{,}566$$

The value of the portfolio is 2,069,161.

(iii) The sum total of probabilities in each row should be 1. The completed matrix of transition probabilities would like this.

		Rating at the end of the year					
		A1	A2	A3	B1	B2	Default
Rating at the beginning of the year	A1	90%	6%	4%	0%	0%	0%
	A2	5%	85%	7.5%	2.5%	0%	0%
	A3	1%	2%	80%	9%	8%	0%
	B1	0%	1%	7.5%	75%	12%	4.5%
	B2	0%	1%	4%	10%	70%	15%

(iv) Out of the various possible combinations the worst that can happen to the portfolio is if A3 rated bond is downgraded to B2 and B2 rated bond defaults.

The likelihood of these rating changes is 8% and 15% respectively.

The joint likelihood of these states (since the events are uncorrelated) is 8%*15% = 1.2%.

(v) The worst outcome lies above the bottom 1% level. Hence the maximum loss sustained by the portfolio in this scenario is the 99% VaR.

The value of the portfolio under this scenario can be calculated as follows.

The value of the A3 rated bond at the end of the first year if its downgrades to B2 is:

$$= 10^6\{0.1 * \{1 + 0.886920 + 0.755784 + 0.582748\} + 1.0 * 0.582748\}$$

$$= 905{,}293$$

The value of the B2 rated bond at the end of the first year if it defaults.

$$= 1{,}000{,}000 * \text{Recovery rate} = 300{,}000$$

Total value of the portfolio $= 1{,}205{,}293$

Therefore the 99% 1-year VaR is equal to

$$= 863{,}868 \text{ (calculated as } 2{,}069{,}161 - 1{,}205{,}293)$$

[Total Marks – 15]

Q2.) **(i)** $\Delta_{Call} = \frac{\partial f}{\partial s} = \Phi(d_1)$

$$\Delta_{Call} \text{ is the delta of a call option on a non} - \text{dividend paying stock}$$

$$\Phi(d_1) = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

S *is the current share price*

T *is the time to maturity*

r *is the riskfree force of interest*

K *is the strike price*

σ *is the volatility parameter*

f is the current value of the call option.

(ii) The stock prices are considered to be "excessively volatile" if the change in market value of stocks (observed volatility), could not be justified by the news arriving. This was claimed to be evidence of market over-reaction which was not compatible with efficiency.

(iii) Using the Garman-Kohlhagen formula we derive the price of the call option at various levels of volatility.

(Students could smartly guess that the volatility is above 10%.)

Volatility	**10.0%**	**20.0%**	**25.0%**	**27.5%**	**30.0%**	**40.0%**
d_1	0.4596	0.2828	0.2581	0.2515	0.2475	0.2475
d_2	0.3889	0.1414	0.0813	0.0571	0.0354	-0.0354
$\Phi(d_1)$	0.6771	0.6114	0.6018	0.5993	0.5977	0.5977
$\Phi(d_2)$	0.6513	0.5562	0.5324	0.5227	0.5141	0.4859
f	4.5027	7.1559	8.5163	9.1990	9.8827	12.6197

Using linear approximation or otherwise students can arrive at a volatility level close to 27.5%.

(iv) Let's assume we buy 'x' units of call option to construct a delta neutral portfolio.

The portfolio now consists of

1,000 units of shares and 'x' units of call options.

The delta of the portfolio is given by

$$\Delta_{Portfolio} = 1{,}000\Delta_{Shares} + x\Delta_{Call}$$

We know that $\Delta_{Shares} = 1$ and $\Delta_{Call} = \Phi(d_1) = 0.5993$

[Using the result in (i) and (iii)]

Substitute the values in the equation and solve for a delta neutral portfolio to derive the units of call option.

$$0 = 1{,}000 * 1 + x * 0.5993$$

x = -1669

So, in order to construct a delta neutral portfolio, the investor needs to sell 1669 call options.

The portfolio is delta neutral but not gamma neutral. That is to say that the delta of the portfolio will change over time. In order to maintain delta neutral position the portfolio will have to be rebalanced on a regular basis. This will increase the transaction cost.

(v) Selling 1669 call options would fetch the investor 15,353 units of cash.

The value of the call option at 5 months to maturity and 10% volatility is

Volatility	**10.0%**
d_1	0.4196
d_2	0.3550
$\Phi(d_1)$	0.6626
$\Phi(d_2)$	0.6387
f	3.9657

The investor will now buy 1,669 call options worth 6,619. Total profit on unwinding the trades is 8,734.

There are a number of assumptions going into this calculation.

- No transaction cost
- No interest earned on cash generated
- No taxes
- No cost of maintaining the margin account
- No bid/offer spread

[Total Marks – 20]

Q3.) Lower bound on a European call option

$$t \text{ is the current time}$$
$$S_t \text{ is the underlying share price at time } t$$
$$T \text{ is the option expiry date}$$
$$r \text{ is the riskfree force of interest}$$
$$c_t \text{ is the price at time } t \text{ of a European call option}$$
$$K \text{ is the strike price}$$

Consider a portfolio, A, consisting of a European call on a non-dividend-paying share and a sum of money equal to $Ke^{-r(T-t)}$.

At time T, portfolio A has a value which is equal to the value of the underlying share, provided the share price S_T is greater than K.

If S_T is less than K then the payoff from portfolio A is greater than that from the share.

Since the option plus cash produces a payoff at least as great as the share, it must have a value greater than or equal to S_t. This gives us a lower bound for c_t :

$$c_t + Ke^{-r(T-t)} \geq S_t$$
$$c_t \geq S_t - Ke^{-r(T-t)}$$

Upper bound on a European call option

A call option gives the holder the right to buy the underlying share for a certain price.

The payoff max{S_T - K,0} is always less than the value of the share at time T,S_T .

Therefore the value of the call option must be less than or equal to the value of the share:

$c_t \leq S_t$

[Total Marks – 4]

Q4.) Under the state price deflator approach we first specify a strictly positive diffusion process A(t) with SDE under P :

$$dA(t) \;=\; A(t)\{\mu_A(t)dt \;+ \sigma_A(t)dW(t)\}$$

where $\mu_A(t)$ and $\sigma_A(t)$ are appropriately chosen stochastic processes and *"P"* denotes a suitably-chosen probability measure.

A(t) is called the state-price deflator and it must be a strictly positive supermartingale.

The following formula is then used to find the price at time *t* of a zero-coupon bond maturing at time *T*:

$$\mathrm{B}(t,T) = \frac{E_P[A(T)|F_T]}{A(t)}$$

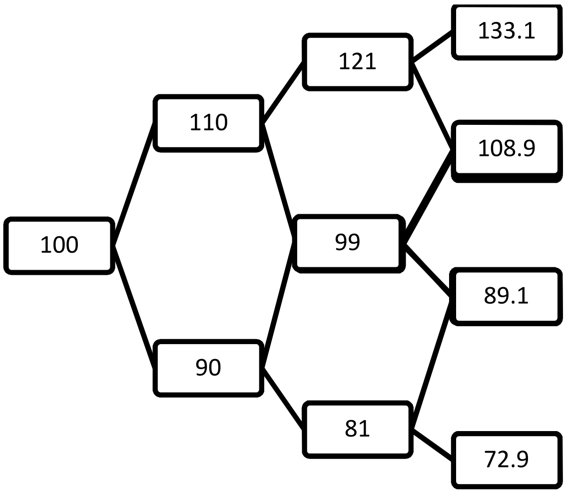
[Total Marks – 4]

Q5.) In order to avoid arbitrage we must have $d \;<\; e^r \;<\; u$

This holds true since $0.9 < \; e^{0.05} < 1.1$.

The sizes of up-steps and down-steps are same in all states. Hence this is a binomial lattice.

The three period binomial tree of share prices will look as under:



The risk neutral probability of an up-step is equal to

$$q = \frac{e^r - d}{u - d} = 0.75636$$

The special option will pay 10 units of cash if the share price is above 120 or below 90 at maturity. Following payoff will be generated at maturity corresponding to the share prices above.

Share price at maturity	Option pay off	Probability
133.1	10	$q^3 = 0.43269$
108.9	0	$3q^2(1-q)^1 = 0.41815$
89.1	10	$3q^1(1-q)^2 = 0.13470$
72.9	10	$(1-q)^3 = 0.01446$

The arbitrage free price of the option can be calculated as follows:

$$V_0 = e^{-3*0.05}[10 * 0.43269 + 0 * 0.41815 + 10 * 0.13470 + 10 * 0.01446 = 5.0080$$

[Total Marks – 7]

Q6.)

(i)
Ans :-

- Investors assume quadratic utility function of wealth
- Investors are risk averse individuals who maximise expected utility of wealth
- Investors all have the same view of asset expected returns

- Asset returns follow a joint normal distribution
- There exists a risk free rate

(ii)

Ans :-

- According to the CAPM the total risk of a risky security is measured by the variance.
- Systematic risk is measured by the covariance of the return on the risky security with the market.
- Only systematic risk is relevant to the pricing of a risky security, the excess of total variance over systematic risk can be diversified away.

(iii) According to the CAPM, the expected return on a single risky stock (or a well diversified portfolio) is given by:

$$R = Rf + \beta(RM\text{-}Rf)$$

Where
Rf is the risk free rate,
RM is the expected market return and
β is the covariance of the return of the risky stock (or well diversified portfolio) with the market, divided by the market variance.

For ABC we have:
$0.085 = Rf + 0.7(RM\text{-}Rf)$, so $0.085 = 0.3Rf + .07RM$

For the 100 share portfolio:
$0.105 = Rf + 1.1(RM\text{-}Rf)$, so $0.105 = \text{-}0.1Rf + 1.1RM$

Which is consistent with $Rf = 0.05$ and $RM = 0.1$

(iv)

Ans :-

- The student s reasoning is flawed because what counts in the determination of expected return is β, not standard deviation (or, equivalently, variance).
- The student believes that the higher standard deviation of ABC (not surprisingly higher than the standard deviation of the 100 stock portfolio,which is well diversified) justifies a higher return for ABC.
- In fact ABC has a lower β than that of the 100 share portfolio and so, Rf and RM being the same for both, the return on ABC should be lower.

[Total Marks – 12]

Q7.)

Ans :-
(i)

- This model is a continuous-time random walk. Graphs of share prices do appear to have this form, with the price changing by a small "random" amount from day to day.
- The RHS contains an *St* factor, which implies that prices changes are proportional to the current price. This is plausible since we would expect price movements to be based on percentage changes, not absolute changes.
- The model assumes that the drift parameter m , reflecting the expected rate of growth, and the volatility parameter s are constant over time. However, the empirical evidence suggests these parameters appear to take different values during different economic eras.
- Attempts to measure the volatility parameter s show that different figures are obtained when s is estimated from daily, weekly or annual price data. (This would suggest that price movements are not independent of the past.)
- The underlying Brownian motion has normal increments. However, studies have shown that the distribution of log-share price increments has flatter tails and is more peaked than a normal distribution.
- Brownian motion assumes independent increments. However the empirical evidence suggests some degree of mean reversion, although this is based largely on a small number of market crashes. In addition, daily movements appear to be subject to "momentum" effects.

Ans :-
(ii)

If $G(S,t) = S^n$ Then $\frac{\partial G}{\partial t} = 0$; $\frac{\partial G}{\partial S} = n\,S^{n-1}$

And $\frac{\partial^2 G}{\partial S^2} = n(n-1)S^{n-2}$

Using Ito's lemma $dG = [\mu nG + n(n-1)\sigma^2 G/2]dt + \sigma nGdz$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is $\mu n + n(n-1)\,\sigma^2/2$

Volatility is $n\sigma$

Ans :-

(iii)

$$\ln S_T \sim \phi[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T]$$

S_T follows lognormal distribution with $E(S_T) = e^{\ln S_0 + \mu - \frac{\sigma^2}{2}T + \frac{\sigma^2}{2}T} = S_0 e^{\mu T}$ and

$$Var(S_T) = e^{2[\ln S_0 + (\mu - \frac{\sigma^2}{2})T] + \sigma^2 T}(e^{\sigma^2 T} - 1) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

(1)

$$E(S_T) = S_0 e^{\mu T} = 50e^{0.16 \times \frac{1}{365}} = Rs.50.022$$

(2)

$$Var(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1) = 50^2 e^{2 \times 0.16 \times \frac{1}{365}}(e^{0.09 \times \frac{1}{365}} - 1)$$

$$Var(S_T) = 0.6171$$

$$SD(S_T) = 0.7855$$

(3)

$$\ln S_T \sim \phi[\ln 50 + (0.16 - \frac{0.3^2}{2}) \times \frac{1}{365}, 0.3^2 \times \frac{1}{365}]$$

$$\ln S_T \sim \phi(3.9123, 0.000247)$$

With 95% confidence,

$$-1.96 < \frac{\ln S_T - 3.9123}{\sqrt{0.000247}} < +1.96$$

$$3.9123 - 1.96 \times 0.0157 < \ln S_T < 3.9123 + 1.96 \times 0.0157$$

$$3.8816 < \ln S_T < 3.9431$$

$$e^{3.8816} < S_T < e^{3.9431}$$

$$48.50 < S_T < 51.58$$

Thus, there is a 95% probability that stock price at the end of one day will lie between 48.50 and 51.58.

[Total Marks – 16]

Q8.)

(i)

- Weak-form hypothesis: stock prices reflect all information that can be derived from studying past market trading data.
- Semi strong: stock prices reflect all publicly available information about the stock.
- Strong form: stock prices reflect all information relevant to the firm, even including information available only to company "insiders".

(ii)

Some examples of valid points are:

- Some managers do appear to generate returns in excess of the market returns on a regular basis.
- This outperformance is not consistent, in particular a manager can not guarantee to produce excess performance in any given year.
- Given the diversity of investment services we would expect by pure chance that some managers would have above average track records over short/medium time periods.

- The outperformance is usually prior to charges being taken into consideration. Once charges are included there is extremely limited evidence of consistent outperformance.
- The risk of positions must be taken into account. A higher risk portfolio should provide, on average, higher returns to compensate for the risk. For sensible comparisons risk adjusted returns are required.
- Fund managers are also employed to build and maintain diversified portfolios or specialist portfolios with specific mandates (ethical or high risk). An efficient market does not mean that tailored portfolios will not be required by some investors.
- Capital markets are closer to the idealised "perfect markets". More likely that inefficiencies arise in the market for buying and selling investment services rather than the markets for buying/selling securities.

(iii)

Multifactor model

The multifactor model attempts to explain returns on assets by relating them to a series of n factors known as indices:

$Ri = ai + bi,1\ I1 + ... + bi,n\ In + ci$ where:
Ri is the return on security i

ai,ci are the constant and random parts of the return, specific to asset i
$I1$,$I2$,...,In are the changes in a set of n indices/factors which explaining the variation of the returns on security i about the expected return ai
bi,k is the sensitivity of the return on stock i to factor/index k
$E[ci]=0$
$cov[ci,cj]=0$ for all $i \neq j$
$cov[ci,Ik]=0$ for all stocks and indices.
The goal of the builders of such a model is to find a set of factors which explain as much as possible of the observed historical variation, without introducing too much noise into predictions of future returns.

(iv)

Three types of factor

1. ***Macroeconomic*** - the factors would include some macroeconomic variables such as interest rates, inflation, economic growth and exchange rates.
2. ***Fundamental*** - the factors will be company specifics such as P/E ratios, liquidity ratios and gearing levels.
3. ***Statistical*** - the factors do not necessarily have a meaningful interpretation. This is because they are derived from historical data, using techniques such as principal components analysis to identify the most appropriate factor

[Total Marks – 12]

Q9.)

Ans :-

(i)

$\text{Var}(X) = 500{,}000^2 * \text{Var}(U) =$

$= 2.5 * 10^{11} * 1/12 =$

$2.08333 * 10^{10}$

(ii) Downside semi-variance of $X = 2.5 * 10^{11}$ * upside semi-variance of U;

the upside semi-variance of U is by symmetry 1/24 so

downside semi-variance of X is $1.04166\ 10^{10}$.

(iii) $P(X < 100{,}000) = P(U > 0.4) = 0.6$

(iv)

Usefulness of downside semi-variance

- It gives more weight to downside risk, *ie* variability of investment returns below the mean, than to upside risk.
- In fact, it completely ignores risk above the mean.
- This is consistent with the investor being risk-neutral above the mean, which is unlikely to be the case in practice.
- The mean is an arbitrary benchmark, which might not be appropriate for the particular investor.
- If investment returns are symmetrically distributed about the mean (as they would be, for example, with a normal distribution) then it will give equivalent results to the variance.
- However, it is much less mathematically tractable than the variance.

[Total Marks – 5]

Q10.)

Ans :-

(a)
Let x be the loss. Hence equivalence of

$E[u(100 - x)] = u[94.5]$

for $u(w) = w + dw^2$ we have

$100 - E(x) + d(100^2 - 2E(x)100 + E(x^2)) = 94.5 + d94.5^2$

$d = (94.5 - 100 + E(x))/(100^2 - 2E(x)100 + E(x^2) - 94.5^2)$

$d = -0.002567$

Ans :-

(b)
For non-satiation
$-\infty < w < -1 / 2d = 194.78$

Therefore utility function cannot help for wealth in excess of 194.78.
Can not use the utility function.

[Total Marks – 5]

[Total Marks – 100]

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