Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2011 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Q.1)

a) To Prove

$$(I\ddot{a})_{\overline{n}|} = (Ia)_{\overline{n}|} + \ddot{a}_{\overline{n+1}|} - (n+1)v^n$$

$$\Rightarrow (I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|} = \ddot{a}_{\overline{n+1}|} - (n+1)v^n$$

$$(I\ddot{a})_{\overline{n}|} - (Ia)_{\overline{n}|} = (1 + 2v + 3v^2 + ... + nv^{n-1}) - (v + 2v^2 + 3v^3 .... + (n-1)v^{n-1} + nv^n)$$

$$= 1 + v + v^2 + ... + v^{n-1} - nv^n$$

$$= 1 + v + v^2 + ... + v^{n-1} + v^n - (n+1)v^n$$

$= \ddot{a}_{\overline{n+1}|} - (n+1)v^n$ (Proved)

(b)

(i)

$$d^{(12)} = 12\left(1 - 1.06^{-(1/12)}\right) = 0.058128$$

$$\ddot{a}^{(12)}_{\overline{25}|} = \frac{(1 - 1.06^{-25})}{0.058128} = 13.19512$$

Price = $1000v^3\ddot{a}^{(12)}_{\overline{25}|}$, where i = 6% p.a.

= 1000 * 11.0789

= 11,078.9

Alternative Solution:

From Tables

$$a_{\overline{25}|} = 12.7834$$

$$\frac{i}{d^{(12)}} = 1.032211$$

$$v^3 = 0.83962$$

Hence, $\ddot{a}^{(12)}_{\overline{25}|} = 12.7834 * 1.032211 = 13.19517$

Price = $1000v^3\ddot{a}^{(12)}_{\overline{25}|} = 1000 * 0.83962 * 13.19517 = 11078.9$

(ii) Price = $1000\ddot{a}_{\overline{3}|}(1+1.05v^{3}+1.05^{2}v^{6}+....+1.05^{9}v^{27})$, where i = 6% p.a.

= $1000\ddot{a}_{\overline{3}|}\ddot{a}_{\overline{10}|}^{@j}$, where j = $1.06^3/1.05 - 1$ = 13.4301%

= 1000 * 2.83339 * 6.05063

= 17,143.80701

(iii) Price = $1050a_{\overline{25}|}-50a_{\overline{5}|}(1+2v^{5}+3v^{10}+4v^{15}+5v^{20})$, where i = 6% p.a.

= 1050*12.78336 -50* 4.21236*7.39778

= 11,864.42

[Total Marks – 11]

Q.2)

(a)

Effective rate of interest (δ) is given by

$$e^{(-20\delta)}=e^{-\int_{0}^{10}(0.005+0.01t)dt}*e^{-\int_{10}^{15}(0.002t)dt}*e^{-\int_{15}^{20}(0.03+0.02t)dt}$$

$$=e^{-\left[0.005t+0.005t^{2}\right]_{0}^{10}}*e^{-\left[0.001t^{2}\right]_{10}^{15}}*e^{-\left[0.03t+0.01t^{2}\right]_{15}^{20}}$$

$$=e^{-0.55}*e^{-0.125}*e^{-1.9}$$

= 0.57695 * 0.8825 * 0.14957

= 0.07615

Therefore δ = 12.87500 %

(b)

The present value of income

$$=\int_{5}^{10}0.5e^{0.155t+0.005t^{2}}e^{-\int_{0}^{t}(0.005+0.01s)ds}dt+100e^{-20\delta}$$

$$= \int_5^{10} 0.5e^{0.155t+0.005t^2} e^{-\left[0.005s+0.005s^2\right]_0^t} dt + 100 * 0.07615$$

$$= \int_5^{10} 0.5e^{0.155t+0.005t^2} e^{-\left(0.005t+0.005t^2\right)} dt + 7.615$$

$$= \int_5^{10} 0.5e^{0.15t} dt + 7.615$$

$$= 0.5 * \left[\frac{e^{0.15t}}{0.15}\right]_5^{10} + 7.615$$

= 7.882 + 7.615

= 15.497

[Total Marks – 10]

Q.3)

a) Returns on fixed interest government bonds may be uncertain due to:

- The coupons will be reinvested on terms, unknown at outset
- Sale price, if sold before redemption, is unknown at outset
- Real return is not known because inflation is not known
- Tax rates may change

b) The difference is between *right* and *obligation.*

Buying a call option costs you money (as you have to pay option premium) and gives you an option whether or not to buy the underlying asset

Selling a put option means you receive money (option premium) and must buy the underlying asset if, and only if, the holder of the option wants to

If a call option is bought, it would be exercised if the market price of the underlying asset is higher than the exercise price.

If a put option is sold, you are likely to be forced to buy the underlying asset if the market price of the underlying asset is lower than the exercise price.

[Total Marks – 6]

Q.4)

Value of the forward contract is given by

$$f = (K_r - K_0)e^{-\delta(T-r)}$$

We have

K_0= 6,800

S_0= 5,000

S_r= 5,800

T = 5 year

r= 1.5 years

D = 3%

We know $K_0 = S_0 e^{(\delta - D)T}$

Thus, $e^{(\delta - D)5} = \dfrac{6800}{5000} \Rightarrow (\delta - D) = \dfrac{1}{5}\ln(\dfrac{6800}{5000})$

$(\delta - 3\%) = 6.150\% \Rightarrow \delta = 9.15\%$

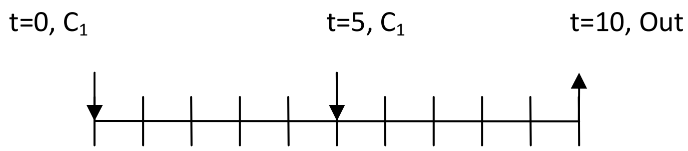
$K_r = S_r e^{(\delta - D)(T-r)} \Rightarrow K_r = 5800e^{6.15\%*3.5} = 7193$

thus, $f = (7193 - 6800)e^{-9.15\%*3.5} = 285.30$

[Total Marks – 4]

Q.5)

(a)



Let C_1 = 9,000 amount invested at start

C_2 = 6,000 amount invested time t = 5

ln(1+i) ~ N(0.1, 0.05) for t < 5

ln(1+i) ~ N(0.1, 0.015) for t ≥ 5.

The accumulation factor S_{n1} for first 5 years follows the distribution

$\ln S_{n1} \sim N(5*0.1, 5*0.05^2)$

Calculation for minimum value of accumulation factor S_{n1} at 99.5% confidence interval at time t= 5 :

We require to find x such that

$$\Pr(\ln(S_{n1}) > \ln x) = 0.995$$

$$\Rightarrow \Pr\left(Z > \frac{\ln x - 0.5}{\sqrt{5} * 0.05}\right) = 0.995$$

Using tables

$$\frac{\ln x - 0.5}{\sqrt{5} * 0.05} = -2.58$$

So, x = $e^{0.5 - 2.58 * \sqrt{5} * 0.05}$

= 1.23559

So minimum accumulated amount at 99.5% confidence interval at time t=5 just after new investment of C_2

9000 * 1.23559 + 6000 = 17,120.31

The accumulation factor S_{n2} for time t=5 to t=10 follows the distribution

$\ln S_{n2} \sim N(5*0.1, 5*0.015^2)$

Calculation for minimum value of accumulation factor S_{n2} at 99.5% confidence interval for time t=5 to t=10 :

We require to find y such that

$$\Pr(\ln(S_{n2}) > \ln y) = 0.995$$

$$\Rightarrow \Pr\left(Z > \frac{\ln y - 0.5}{\sqrt{5} * 0.015}\right) = 0.995$$

Using tables

$$\frac{\ln y - 0.5}{\sqrt{5} * 0.015} = -2.58$$

So, y = $e^{0.5-2.58*\sqrt{5}*0.015}$

= 1.51205

So minimum accumulated amount at 99.5% confidence interval at time t=10 is

= 17,120.31* 1.51205

= 25,886.76

(b) Guaranteed maturity outgo

= 9000* 1.095^{10} + 6000* 1.095^{5}

= 31,749.48

Possible loss amount at 99.5% confidence level

= 31,749.48 - 25,886.76

= 5862.72

The amount required at time t = 0 in order to mitigate the loss at 99.5% confidence interval is

= 5862.72* 1.06^{-10}

= 3273.71

[Total Marks – 12]

Q.6)

(a)

Effective duration of assets

$$= \frac{(2*150v^3 + 5*15v^6 + 6*20v^7 + 7*62v^8)}{(150v^2 + 15v^5 + 20v^6 + 62v^7)}$$, where i=8% p.a.

$$= \frac{589.9079}{187.5894}$$

= 3.1447

Effective duration of liabilities

$$= \frac{(34v^2 + 3*140v^4 + 5*31.5v^6 + 7*40.331v^8)}{(34v \;\; + 140v^3 + 31.5v^5 + 40.331v^7)}$$, where i=8% p.a.

$$= \frac{589.6409}{187.5891}$$

= 3.1433

(b) With 1% increase in discount rate

Present value of assets

= $150v^2 + 15v^5 + 20v^6 + 62v^7$, where i=9% p.a.

= 181.8424

Present value of liabilities

= $34v \;\; + 140v^3 + 31.5v^5 + 40.331v^7$, where i=9% p.a.

= 181.8336

Thus with 1% increase of interest rate PV of Assets $\approx$ PV of Liabilities

[Total Marks – 7]

Q.7)

a)

The inflation rates implied by the table of indices are:

Year	2007-08	2008-09	2009-10	2010-11
Inflation	5.70%	4.79%	8.57%	21.05%

b)

Coupon and redemption payments

Coupon payment made in April 2009 = 8.25*167/158 = 8.72
Coupon payment made in April 2010 = 8.25*175/158 = 9.14
Coupon payment made in April 2011 = 8.25*190/158 = 9.92 and
Redemption payment made in April 2011 = 100*190/158 = 120.25

Year	2009	2010	2011	2011
Payments	8.72	9.14	9.92	120.25

c)

Real value of coupon and redemption payments

Real value of coupon payment made in April 2009 = 8.72*167/175 = 8.32
Real value of coupon payment made in April 2010 = 9.14*167/190 = 8.03
Real value of coupon payment made in April 2011 = 9.92*167/230 = 7.20 and
Real value Redemption payment made in April 2011 = 120.25*167/230 = 87.31

Year	2009	2010	2011	2011
Real Payments	8.32	8.03	7.20	87.31

d)

No. From a), we can see that inflation has been fairly stable from year 2007 to 2009, but then it leapt to 21% in the last year. Because of the time lag of one year in applying the indices, there is no inflation protection in the last year and thus the real value of the last coupon payment is much lower than that of other coupon payments. Due to the same reason, the real value of the redemption amount has also fallen substantially.

e)

Price P = $8.72v + 9.14v^2 + 9.92v^3 + 120.25v^3$ @7%p.a. effective

= 122.39

[Total Marks – 13]

Q.8)

(a)

Annual effective rate of interest

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = 9.84383\%$$

Required Equal Annual Installment, X is given by,

$Xa_{\overline{9|}}^{@\,9.84383\%\,pa}$ =1,000,000 ; $a_{\overline{9|}}^{@\,9.84383\%\,pa}$ =5.794948

X =1,000,000/5.794948 = 172,564

(b) Loan outstanding at the end of 2nd year

= 172,564 $a_{\overline{7|}}^{@\,9.84383\%\,pa}$ = 844,451

Additional loan capital Mr. Preferred Customer will pay at the end of 2nd year

$= Min\left(500{,}000 * \left(1+\frac{0.06}{2}\right), 30\% * 1{,}000{,}000\right)$ = 300,000

Remaining amount with Mr. Preferred Customer

$= 500{,}000 * \left(1+\frac{0.06}{2}\right) - 300{,}000 =$ 215,000

Therefore loan outstanding at the end of 2nd year after pre-payment

= 844,451 – 300,000 = 544,451

Loan capital repaid in 3rd year from annual installment

= 172,564- 544,451*9.84383%

= 118,969

Additional loan capital Mr. Preferred Customer will pay at the end of 3rd year

$= Min\left(215{,}000 * \left(1+\frac{0.06}{2}\right)^2, 30\% * 1{,}000{,}000\right)$ = 228,094

Therefore loan outstanding at the end of 3rd year after pre-payment

= 544,451 - 118,969 - 228,094

= 197,389

[Total Marks – 9]

Q.9)

(a)

(i) **DPP for Type 1** is found by solving the following equation for least value of n

$$-100{,}000+\frac{14{,}402}{12}a_{\overline{n}|}^{0.5\%}>0$$ where n is the number of months

i.e., $a_{\overline{n}|} > 83.32176$

=> $1-v^n > 0.416609$

=> $v^n < 0.583391$

=> $-n * \ln(1+i) < \ln(0.583391)$

=> $n > -\ln(0.583391)/ \ln(1+i)$

=> n> 108.0487 or 9.0041 years

=> **DPP(Type 1) = 109 months or 9 years and 1 month**

DPP for Type 2 is found by solving the following equation for least value of n

$$-100{,}000+\frac{11{,}400}{12}a_{\overline{n}|}^{0.5\%}>0$$ where n is the number of months assuming n<180 so that capital repayment is not taken into consideration (i.e., assuming that DPP is less than the full term)

i.e., $a_{\overline{n}|} > 105.2632$

=> $1-v^n > 0.526316$

=> $v^n < 0.473684$

=> $-n * \ln(1+i) < \ln(0.473684)$

=> $n > -\ln(0.473684)/ \ln(1+i)$

=> n> 149.8162 or 12.4847 years

=> **DPP(Type 2) = 150 months or 12 years and 6 months**

(ii) NPV for Type 1 with monthly i = 0.5% is given by

$$NPV_1=-100{,}000+\frac{14{,}402}{12}a_{\overline{180}|}^{0.5\%}=42{,}224$$

NPV for Type 2 with monthly i = 0.5% is given by

$$NPV_2=-100{,}000+\frac{11{,}400}{12}a_{\overline{180}|}^{0.5\%}+100{,}000v^{180\,@\,0.5\%}=53{,}327$$

(iii) Both products are profitable for investor. Though DPP for Type 1 is lower than Type 2, but NPV for Type 2 is higher than Type 1 and hence, he should invest in Type 2 as he is able to borrow money from market and hence, DPP should not be a criterion.

(b)

Money Weighted Rate of Return, i, should satisfy following equation of value

$$300(1+i)^5 + 1500(1+i)^{4.5} + 500(1+i)^4 - 700(1+i)^{2.5} + 200(1+i) = 2624$$

Calculation for initial approximation for i, (using approximation $(1+i)^n \approx (1+ni)$)

$$300(1+5i) + 1500(1+4.5i) + 500(1+4i) - 700(1+2.5i) + 200(1+i) = 2624$$

$$\Rightarrow (300+1500+500-700+200) + (1500+6750+2000-1750+200)i = 2624$$

$$\Rightarrow 1800 + 8700i = 2624$$

$$\Rightarrow i = \frac{824}{8700}$$

$$\Rightarrow i = 9.47\%$$

Trial for required level of accuracy of i
using i = 9.5% LHS = 2,788
using i = 9% LHS = 2,728
using i = 8% LHS = 2,609
So i should be slightly greater than 8%
Using interpolation between 8% and 9%,

$$\frac{i-8\%}{9\%-8\%} = \frac{2624-2609}{2728-2609}$$

=> i = 8.13%
Hence required Money Weighted Rate of Return is 8.1% to the nearest .1%

Additional information required to calculate Time Weighted Rate of Return is the value of fund just before each of the given cash-flows.

[Total Marks – 16]

Q.10)

Here coupon income per unit D = 8

Redemption amount R = 105

(1-income tax rate)g = (1-income tax rate)D/R = 6.85714%

(assuming tax is payable immediately; deferred tax will increase this value)

Rate of interest charged by bank, payable quarterly

$$= i^{(4)} = 4[(1+i)^{(1/4)} - 1] = 6.34731\%$$

Thus as (1-income tax rate)g > $i^{(4)}$

and further, deferment of income tax will increase the value of left hand side, Mr. Intelligent will opt for maximum redemption term.

Maximum profit that Mr. Intelligent can make

$$= 100*(8a_{\overline{15|}}^{@\, i^{(4)}} + 105v^{15} - 100 - 8*10\%v^{1.5}a_{\overline{15|}}^{@\, i^{(4)}}),$$

Where, $a_{\overline{15|}}^{@\, i^{(4)}} = \dfrac{1-(1+.065)^{-15}}{0.0634731}$ =9.62885

= 1,084.89

[Total Marks – 5]

Q.11)

a)

Using the formula

$$P_t = \frac{1}{(1+y_t)^t}$$

$$\Rightarrow y_t = \left(\frac{1}{P_t}\right)^{(1/t)} - 1$$, we get

$$y_5 = \left(\frac{1}{0.59790}\right)^{(1/5)} - 1 = 10.83\%$$

b)

Amount of money Mr. A will have at time t=5 out of his investment at time t=0 is

$$= 75{,}000*(1+y_5)^5 = \frac{75{,}000}{P_5} = \frac{75{,}000}{0.59790} = 1{,}25{,}439.04$$

Total amount required at time t=5 to purchase required bonds

$$= 20{,}000\left(1 + f_{5,2}\right)^{-2} + 20{,}000\left(1 + f_{5,3}\right)^{-3} + 20{,}000\left(1 + f_{5,4}\right)^{-4} + 1{,}05{,}000\left(1 + f_{5,5}\right)^{-5}$$

$$= 20{,}000\frac{\left(1 + y_7\right)^{-7}}{\left(1 + y_5\right)^{-5}} + 20{,}000\frac{\left(1 + y_8\right)^{-8}}{\left(1 + y_5\right)^{-5}} + 20{,}000\frac{\left(1 + y_9\right)^{-9}}{\left(1 + y_5\right)^{-5}} + 1{,}05{,}000\frac{\left(1 + y_{10}\right)^{-10}}{\left(1 + y_5\right)^{-5}}$$

$$= 20{,}000\frac{P_7}{P_5} + 20{,}000\frac{P_8}{P_5} + 20{,}000\frac{P_9}{P_5} + 1{,}05{,}000\frac{P_{10}}{P_5}$$

$$= 18{,}287.34 + 17{,}959.53 + 17{,}835.76 + 93{,}672.85$$

$$= 1{,}47{,}755.48$$

Therefore additional capital required at time t=5 is

$$= 1{,}47{,}755.48 - 1{,}25{,}439.04 = 22{,}316.44$$

[Total Marks – 7]

[Total Marks – 100]

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