Institute of Actuaries of India

Subject CT4 – Models

May 2010 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Soln 1.

A stochastic process is said to be strictly stationary if the joint distributions of $X_{t_1}, X_{t_2}, \ldots X_{t_n}$ and $X_{k+t_1}, X_{k+t_2}, \ldots X_{k+t_n}$ are identical for all $t_1, t_2, \ldots t_n$ and $k+t_1, k+t_2, \ldots k+t_n$ in J and in all integers n. The statistical properties of the process, such as mean, variances etc, remain identical as time elapses.

A process is said to be weakly stationary if:

- $E(X_t)$ is constant for all t and
- $Cov(X_t, X_{t+k})$ depends only on lag k

[3]

Soln 2.

(a) We will have following problems due to duplicate policies.

- The death of an individual with multiple life insurance policies would have a disproportionately large impact on the crude mortality rates.
- It increases the variance of the mortality rates.
- It invalidates many of the graduation tests as the tests assume that recorded deaths are statistically independent.

(b) Number of policies = $(1\times 2214) + (2\times 124) + (3\times 32) + (4\times 0) + (5\times 4)$
= 2578

The expected number of policies becoming claims by death:

E[c] = $\sum_i i\, \pi_i$ N q_x

= 2578 $\times$ .00498 = 12.84

The variance = V[c] = $\sum_i i^2\, \pi_i$ N q_x (1- q_x)

= (2214+496+288+100) $\times$(1-.00498)*.00498
= 15.35

[5]

Soln 3.

(a) The probability that a sick life becomes terminally sick when it leaves the sick state is 0.2 / (0.2+0.2+0.8) = 0.167

(b) Expected Holding Time = 1/(0.2+0.2+0.8) = 1/1.2 =0.833 years.

(c) Define: t_i as the expected future lifetime given that the life is in status *i* presently, which means t_H is expected future life time of healthy life, t_S is expected future life time of sick life and t_T is expected future life time of terminally sick life.

We have:

$$t_H = \frac{1}{0.04 + 0.01} + \frac{0.04}{0.04 + 0.01} t_S$$

$$\therefore t_H = 20 + 0.8 t_S \qquad \text{......(i)}$$

Similarly:

$$tS = \frac{1}{0.8+0.2+0.2} + \frac{0.8}{0.8+0.2+0.2} tH + \frac{0.2}{0.8+0.2+0.2} tT$$

$$t_T = 2$$

$$tS = \frac{7}{6} + \frac{4}{6} tH \qquad \text{..........(ii)}$$

Solving the above equations (i) & (ii) , we get:

t_H = 44.8571 years and t_S= 31.0714 years.

[7]

Soln 4.

(a) The low volume of data is the principal problem in the given situation, hence;

- The crude rates may not be suitable for the purpose.
- There are random sampling errors
- The rate at a particular age can not be linked to the rates at adjacent ages
- Rates will not progress smoothly from age to age which allows a practical smooth set of premium rates to be produced

(b) (Need not write all the columns, as given in the question, as those are not required)

Age	z_x	z_x^2	$\mathring{\mu}_{x+1/2}$	$\Delta \mathring{\mu}_{x+1/2}$	$\Delta^2 \mathring{\mu}_{x+1/2}$	$\Delta^3 \mathring{\mu}_{x+1/2}$
18-22	0.5249	0.27552	0.0061	0.0070	0.0061	0.0033
23-27	0.3615	0.13068	0.0131	0.0131	0.0094	0.0033
28-32	1.0266	1.05391	0.0262	0.0225	0.0127	0.0020
33-37	-1.0912	1.19072	0.0487	0.0352	0.0147	-0.0009
38-42	1.2394	1.53611	0.0839	0.0499	0.0138	-0.0044
43-47	0.5949	0.35391	0.1338	0.0637	0.0094	-0.0076
48-52	0.6346	0.40271	0.1975	0.0731	0.0018	
53-57	0.1787	0.03193	0.2706	0.0749		
58+	-0.0367	0.001349	0.3455			
		4.976839				

Test Statistics: $X = \sum z_i^2$

<u>Null Hypothesis:</u> H_0: X has a Chi Square distribution

The observed value of X is 4.98

The degrees of freedom are equal to the number of age groups less some allowances for the method of graduation. So, degree of freedom is 9 (Max).

This is a one sided test. Upper 5% point of Chi Square distribution exceeds the observed value 4.98 for all degrees of freedom (<= 9) except one. There is insufficient evidence to reject H_0.

The third difference of $\overset{o}{\mu}_{x+1/2}$ is not very insignificant and hence the graduated rates are not very smooth.

[8]

<u>Soln 5.</u>

(a) Once the process goes into state 3, it is perpetually locked in state 3. The minimum value of Z, therefore, is $^1/_6$, the starting distribution in state 3.

(b) Once the process goes into state 3, the process stays at state 3 for ever. Therefore, the transition probabilities relating to state 3 are:

$$p_{31}(t) = 0$$
$$p_{32}(t) = 0$$
$$p_{33}(t) = 1$$

The process can never reach state 1 when it starts out in state 2. Therefore;

$p_{21}(t) = 0$

Once the process goes out of state 2, it can never return to state 2 again. Therefore:

$p_{22}(t) = p_{\overline{22}}(t) = e^{-t}$

Since the transition probabilities sum to 1, we have:

$p_{23}(t) = 1 - e^{-t}$

Once the process goes out of state 1, it can never return to state 1 again. Therefore:

$p_{11}(t) = p_{\overline{11}}(t) = e^{-4t}$

In order to determine the transition probabilities from state 1 to state 2, we shall need to write down and solve Chapman-Kolmogorov differential equations:

The forward differential equation for $p_{12}(t)$ is:

$$\frac{d}{dt}p_{12}(t) = p_{11}(t)\mu_{12} + p_{12}(t)\mu_{22} + p_{13}(t)\mu_{32}$$

Putting the transition rates:

$$\frac{d}{dt}p_{12}(t) = 3p_{11}(t) - p_{12}(t)$$

Substituting the value for $p_{11}(t)$ and taking $p_{21}(t)$ to the other side:

$$\frac{d}{dt}p_{12}(t) + p_{12}(t) = 3e^{-4t}$$

By integrating factor method, the integrating factor is e^t. Therefore:

$$\frac{d}{dt}\{e^t p_{12}(t)\} = 3e^{-3t}$$

Integrating w.r.t. t, we have:

$$e^t p_{12}(t) = -e^{-3t} + c$$

$$\therefore p_{12}(t) = -e^{-4t} + ce^{-t}$$

We know that $p_{12}(0) = 0$. Therefore, the constant of integration works out to be 1.

$$\therefore p_{12}(t) = e^{-t}(1 - e^{-3t})$$

Since the transition probabilities must sum to one, we have:

$$p_{13}(t) = 1 - p_{11}(t) - p_{12}(t)$$

$$\therefore p_{13}(t) = 1 - e^{-t}$$

Therefore, the transition matrix is given as follows:

$$\begin{bmatrix} e^{-4t} & e^{-t}(1 - e^{-3t}) & 1 - e^{-t} \\ 0 & e^{-t} & 1 - e^{-t} \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the distribution after time 1 is:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} e^{-4} & e^{-1}(1 - e^{-3}) & 1 - e^{-1} \\ 0 & e^{-1} & 1 - e^{-1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.009158 & 0.297408 & 0.693434 \end{bmatrix}$$

[9]

<u>Soln 6.</u>

(a) <u>Type – I Censoring:</u>

If the censoring times are known in advance (a degenerate case of random censoring), then the mechanism is called type – I censoring.

<u>Type – II Censoring:</u>

If observation is continued until a predetermined number of deaths have occurred then Type – II censoring is said to be present.

(b)

(i) Usual decrements:

- Normal age retirement
- Resignation
- Death
- Transfer to any other pension scheme

Type – I censoring is present as the selected individuals are stopped to be observed once they have reached their 60th birth day.

Type – II censoring is not present as there is no predetermined number of deaths in the investigation.

(ii) The original observations can be reordered and written as:

1(*), 3(*), 5(*), 6, 6, 8(*), 10(*), 12, 15, 18 (*), 20, 20, 23

The following table shows the calculations:

Time t_j	c_j	d_j	n_j	d_j/n_j	$\sum_1^j (d_j/n_j) = \Lambda_t$
$0 \le t < 6$	3	0	50	0	0
$6 \le t < 12$	2	2	47	0.04255	0.04255
$12 \le t < 15$	0	1	43	0.02326	0.06581
$15 \le t < 20$	1	1	42	0.02381	0.08962
$20 \le t < 23$	0	2	40	0.05000	0.13962
$t \ge 23$	0	1	38	0.02632	0.16594

[11]

<u>Soln 7.</u>

(a) The probability density function of $F_x(t)$ is given by:

$f_x(t) = \frac{d}{dt}(F_x(t))$, where $F_x(t)$ = P [T_x <= t]

$$f_x(t) = \lim_{h->0+} (1/h)[\ P\ (T_x <= t+h) - P\ (T_x <= t)]$$

$$= \lim_{h->0+} (1/h)[\ P\ (T<= x+t+h/T> x) - P\ (T<= x+t/T>x)]$$

$$= \lim_{h->0+} [\ P\ (T<= x+t+h) – P[T<=x] -P\ (T<= x+t]+ P[T<=x]/[h*S(x)],$$

Where S(x) = P [T>x],

Multiplying and dividing by S (x+t), we get

$$f_x(t) = [S(x+t)/S(x)]* \lim_{h->0+} [\ P\ (T<= x+t+h) - P\ (T<= x+t)]/[h*S(x+t)]$$

$$= S_x(t) * \lim_{h->0+} [P\ (T<= x+t+h/\ T>x+t)]/h$$

$$= S_x(t) * \mu_{x+t} = {}_t p_x * \mu_{x+t} \quad \text{(for 0<=t<w-x)}$$

(b) The probability density function $f_x(t)$, by definition, denotes that the probability of a life aged x will die between age x+t and x+t+dt is $f_x(t)$ dt,

Which is nothing but the probability that the life with exact age x will survive upto age x+t $\times$ the probability that the life will die within the period dt which is ${}_t p_x \times {}_{dt} q_{x+t}$

That is, ${}_t p_x * \mu_{x+t}$ *dt using the approximate relationship: ${}_{dt} q_{x+t} = \mu_{x+t}$ *dt

Therefore, $f_x(t) = {}_t p_x * \mu_{x+t}$

(c)

(i) K is the limiting age and t lies between x and K-x.

(1/2 mark for each fact, max 1)

(ii) $\mu_{x+t} = -\frac{\partial}{\partial t}(\ln(S_x(t))$

Replacing the value of $S_x(t)$,

$\mu_{x+t} = -\frac{\partial}{\partial t}[\ln(K-x-t)-\ln(K-x)]$

$= \frac{1}{K-x-t}$, where $0 \le t < K-x$

(iii) Given $\mu_x = \frac{1}{100-x}$; so ${}_t p_x = \frac{100-x-t}{100-x}$, from the results above.

The probability that a life aged 20 exact will die between exact age 40 and 45

$= {}_{10}P_{20} * {}_{10}P_{30} * {}_5 q_{40}$

$= {}_{10}P_{20} * {}_{10}P_{30} * (1- {}_5P_{40})$

Now, ${}_{10}P_{20} = e^{\int_0^{10}(-.025)*ds} = e^{-.025*10} = 0.77880$

${}_{10}P_{30} = \frac{100-30-10}{100-30} = 0.857143$

${}_5P_{40} = \frac{100-40-5}{100-40} = 0.91667$

The required probability = 0.97531*0.857143*(1-0.91667) = 0.0556

[12]

Soln 8.

(a) The key objective of the model is to calculate the *optimal* bid amount.

(b) *A good solution should cover the following points.*

- Cash flow projection model
- Should cover cash flows such as construction costs, projected toll revenues, projected maintenance expenses etc.
- The bid amount shall be the expected construction costs and maintenance expenses adjusted to reflect the expected toll revenues.
- Data on vehicular traffic from the other such bridges in the city may be used to have an indicative idea of toll revenues.
- Allowance for expense inflation should be reflected in the model.
- Allowance for time value of money may be reflected by a suitable risk discount rate.
- A suitable allowance for taxation should be made. Consider any preferential tax treatment given to infrastructure projects.

(c) *List of the parameters:*

(i) Construction expenses per km
(ii) Estimated vehicular traffic
(iii) Mix of vehicular traffic, if differential rates are charged to different vehicles
(iv) Growth rate in vehicular traffic
(v) Estimated maintenance expenses
(vi) Maintenance expense inflation
(vii) Risk discount rate, reflecting cost of capital
(viii) Tax rates

(d) *Possible approaches are:*

(i) Compare with bids for similar contracts awarded recently
(ii) Perform sensitivity tests on the parameters chosen.
(iii) Involve real world experts and have their views.

(e) The key idea here is sensitivity testing. One must vary the parameters in the model and analyse the impact of the variation on the bid amounts. For example, a 5% change in maintenance expenses may result in 20% change in the bid amount due to the long duration of the contract whereas a 5% change in construction costs may result in, say, only 10% change in the bid amount. Such sensitivity analysis helps identifying where risks lie: Mis-estimating such parameters may prove very costly.

[12]

<u>Soln 9.</u>

(a) Relationship is:

$$P_x(t) = \frac{1}{2}[p'_x(t) + p'_{x+1}(t)]$$

- P_x(t) denotes the number of lives at time t aged x last birth day. Hence this will comprise of lives between aged x and x+1.
- Lives who are between age x and x+1/2 are aged x nearest birthday.
- Lives who are between ages x+1/2 and x+1 are aged x+1 nearest birthday.
- So, P_x(t) can be approximately the average of the lives who are aged x nearest birthday and the lives who are aged x+1 nearest birthday.

  Hence, $P_x(t) = \frac{1}{2}[p'_x(t) + p'_{x+1}(t)]$

  Assumption: The birthdays are uniformly distributed over the calendar year.

(b) Deaths are classified according to age last birthday. Therefore, life year rate interval is used.

The initial mortality (q) = $\frac{d_x}{E_x}$ and the force of mortality (μ) = $\frac{d_x}{E_x^c}$

q estimates q_x and μ estimates $\mu_{x+1/2}$

d_x = Total number of deaths aged x last birth day during investigation.

$$= \sum_{t=K}^{K+N-1} d_x(t)$$

$$E_x = E_x^c + (1/2)^* d_x$$

Assumption: Deaths are uniformly distributed over each year of age

$$E_x^c = \int_0^N P_x(t)\,dt$$

In view of the data given, the values of Px (t) are provided for Px (1/2), $Px\,(1\frac{1}{2})$ etc. The integral should be evaluated to make convenient of the use of the data given.

Assumption: Px (t) moves linearly within each calendar year (t).

Therefore, E_x^c can be approximated by $\sum_{t=0}^{N-1} Px\,(t+\frac{1}{2})$

$\mu_{x+1/2} = \dfrac{d_x}{E_x^c}$ and $q_x = \dfrac{d_x}{E_x}$

(c) The candidate is supposed to show one sample calculation as below, else deduct 1 $\frac{1}{2}$ marks.

When age, x = 63 yrs

E_x^c = 4214+4535+4751 = 13500 and d_x = 49+55+58 = 162

$E_x = E_x^c$ + (1/2)* d_x = 13581

Age	E_x^c	d_x	E_x	$\mu_{x+1/2}$	q_x
63	13500	162	13581	.0120	.0119
64	12800	160	12880	.0125	.0124
65	12500	165	12583	.0132	.0131

[14]

Soln 10.

(a) The process may be expressed as a Markov chain by considering following states. Each state indicates the current number of successive defeats.

State	Number of successive defeats
1	0 – Not defeated last time
2	1 – Defeated in the last match
3	2 – Defeated in the last two matches
4	3 – Defeated in the last three matches
5	4 – Captain sacked

The transition matrix is as follows:

$$\begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) A Markov chain is said to be irreducible if any state *j* can be reached from any state *i*. The above process is not irreducible as the captain, once sacked, can never become the captain again.

(c)

(i) The probability of remaining the captain for exactly four matches is given by:

Match #	#1		#2	#3	#4	Probability
Result	Lose		Lose	Lose	Lose	
Probability	0.3	0.3	0.3	0.3	=0.3^4 = 0.0081	

(ii) The probability of remaining the captain for exactly five matches is given by:

Match #	#1	#2	#3	#4	#5	Probability
Result	Not Lose	Lose	Lose	Lose	Lose	
Probability	0.7	0.3	0.3	0.3	0.3	0.00567

(iii) The probability of remaining the captain for exactly seven matches is given by:

Match #	#1	#2	#3	#4	#5	#6	#7	Probability
Result	Any	Any	Not Lose	Lose	Lose	Lose	Lose	
Probability	1	1	0.7	0.3	0.3	0.3	0.3	0.00567

(iv) The probability of remaining the captain for exactly nine matches is given by:

Match #	#1 #2 #3 #4	#5	#6	#7	#8	#9	Probability
Result	Not all four losses	Not Lose	Lose	Lose	Lose	Lose	
Probability	=1-0.3^4=.9919	0.7	0.3	0.3	0.3	0.3	0.0056241

(d) Define N_i = Expected number of matches as a captain given that the current state is i. Since the captain is newly appointed, the variable N in the question is N_1 as defined here.

We have:

$N_1 = 1 + 0.7 \times N_1 + 0.3 \times N_2$

$N_2 = 1 + 0.7 \times N_1 + 0.3 \times N_3$

$N_3 = 1 + 0.7 \times N_1 + 0.3 \times N_4$

$N_4 = 1 + 0.7 \times N_1$

Solving the above equations; we get:

$N_4 = 123.46$

$N_3 = 160.49$

$N_2 = 171.60$ and

$N_1 = 174.94$

Therefore, E(N) is 174.94 matches.

[19]

[Total Mark 100]

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