Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2010 Examinations

INDICATIVE SOLUTIONS

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1 (i) (a)

<u>Cost</u>:-

Weather Scenario& Probability	Strike Scenario & Probability	Combined probability	PV of Cost @ 10%	Expected PV of cost PV of Cost* Probability
Good Weather(2/3)	No Strike (1/2)	2/6	72	24
Good Weather(2/3)	Strike (1/2)	2/6	72*1.2	28.80
Bad Weather(1/3)	No Strike(3/4)	3/12	72+36v	26.18
Bad Weather(1/3)	Strike(1/4)	1/12	(72+36v)*1.2	10.47
			Total	89.45

Check : Total probability should be =1 (2/6 + 2/6 + 3/12 + 1/12 =1)

Expected PV of cost = 89.45 crores

<u>Revenue:</u>

Expected PV(Revenue) = Prob(Success).PV(Revenue/Success) + Prob(Failure).PV(Revenue/Failure)

$= 4/5 * 60 * \ddot{a}_{\overline{3}|} v^2 + 1/5 * 30 * \ddot{a}_{\overline{3}|} v^2$ @ 10%

$= (48+6) * \ddot{a}_{\overline{3}|} v^2$ @ 10%

= 54 * 2.260775 = 122.0818 crores

NPV = EPV(Revenue) – EPV(Cost)

= 122.0818 – 89.45 = 32.63 crores

b) Prob(Min Cost) = 2/6

Prob(Max Revenue) = 4/5

$\therefore$ Prob(Best Scenario) = (2/6)*(4/5)=8/30=4/15

[10]

2. (i) $P = 75v+75*1.03v^2+75*1.03^2v^3+\ldots\ldots + 75*1.03^{19}v^{20} + 1050\, v^{20}$ @ 8.25%

$= 75v(1+ 1.03v + (1.03v)^2 + (1.03v)^3 + \ldots\ldots + (1.03v)^{19}) + 1050\, v^{20}$ @ 8.25%

$=(75/1.0825)\, \ddot{a}_{\overline{20}|\, j}$@ 5.0971% + $1050\, v^{20}$ @ 8.25%

= 900.0177 + 215.0954 = 1115.113

$\therefore$ P = 1115.11

(ii) To find n:

$195.5 = 20\, a_{\overline{n}|} + 100v^n$ @ 8.5%

$= \frac{20}{0.085} - \frac{20}{0.085} v^n + 100\, v^n$

$v^n = 39.79412/135.2941 = 0.29413$

n = ln(0.29413)/ln(v) where ln = natural logarithm

n = 15.00039 ~ 15 years

When there is a possibility of default, we have to multiply each payment with the probability

of payment.
Discounting factor of t^{th} payment = $(0.94017)^t v^t$ @ 10% = $(0.8547)^t = (1/1.17)^t$
Hence, calculate PV @ 17% which will be equivalent to multiplying by probability of default.

PV of payment streams = $20\, a_{\overline{15}|} + 100v^{15}$ @ 17%
= 115.9726

[10]

3. (i) Let PV(Annuity 1)=A & PV(Annuity 2)=B

Then, $A = 10v + 9v^2 + 8v^3 + \ldots\ldots + 2v^9 + 1v^{10}$ @ i
$A(1+i) = 10 + 9v + 8v^2 + \ldots\ldots + 2v^8 + 1v^9$
Subtracting, we get
$Ai = 10 - (v+v^2+v^3+\ldots\ldots\ldots+v^9+v^{10}) = 10 - a_{\overline{10}|}$

$$A = \frac{10 - a_{\overline{10}|}}{i} \quad \ldots\ldots\ldots\ldots\ldots \ (1)$$

$B = v + 2v^2 + 3v^3 + \ldots\ldots + 10v^{10} + 11(v^{11} + v^{12} + v^{13} + v^{14} + \ldots\ldots\ldots)$ @ i
$= (Ia)_{\overline{10}|} + 11\, v^{10}\, a_{\overline{\infty}|} = (Ia)_{\overline{10}|} + \frac{11\, v^{10}}{i}\ldots \quad \ldots\ldots\ldots\ldots\ldots \ (2)$

(1)+(2) gives $A+B = \frac{10 - a_{\overline{10}|}}{i} + (Ia)_{\overline{10}|} + \frac{11\, v^{10}}{i}$

Given 2A = B. Thus A+B = A+2A = 3A

Hence, $A+B = 3A = \frac{10 - a_{\overline{10}|}}{i} + (Ia)_{\overline{10}|} + \frac{11\, v^{10}}{i}$

$\Rightarrow 3Ai = 10 - a_{\overline{10}|} + \ddot{a}_{\overline{10}|} - 10v^{10} + 11v^{10}$
$= 10 - v^{10} + 1 + v^{10} = 11$

<u>Hence A = (11/3i)</u> $\ldots\ldots\ldots\ldots\ldots$ (3)

From (1) and (3), $10 - a_{\overline{10}|} = 11/3$
$\Rightarrow a_{\overline{10}|} = 10 - 11/3 = 19/3 = 6.3333$
From tables, at 10% $a_{\overline{10}|} = 6.1446$
At 9% $a_{\overline{10}|} = 6.4177$

<u>By interpolation i = 0.09309</u>
Substituting this value in (3), we get A = 11/(3*0.09309)=39.3888 ~ 39.4

<u>PV(Annuity 1) = 39.4</u>

[10]

4. i) a) $i^{(12)} = 7.75\%$
$i^{(12)}/12 = 0.645833\%$
Capital content of t^{th} installment = (Installment amount)* $v^{(12n-t+1)}$ @ 0.645833%

Capital content of 16th installment = $14612.884v^{(12n-15)}$=3433.056
$v^{(12n-15)}$=0.2349
12n-15=ln(0.2349)/ln(v)
=> 12n=239.3999 ~ 240 months or 20 years
n=20 years

Y = $14612.884a_{\overline{240|}}$ @ 0.645833%
= 14612.884 * 121.8107 =17,80,000

Hence, Y = Rs.17,80,000

b) Loan o/s after 36 installments = 14612.884 $a_{\overline{204|}}$ @ 0.645833%
= 14612.884*113.1963= 1654124.187
Loan o/s after lump-sum payment of 300,000=1354124.187
Interest rate applicable = 7.5% p.a. effective

Let New EMI=Q
Then $12Qa^{(12)}_{\overline{17|}}$ =1354124.187 @ 7.5%
Q = 1354124.187/(12*9.754108) =138827.3/12
Q=11568.94
Revised EMI = 11568.94

ii) Intt content if I repay in one lumpsum = $X[1.065^{10}-1]$
Intt content if I repay through 10 level installments = [Installment*10-X]
Where Installment =$X/a_{\overline{10|}}$
Given, $X[1.065^{10}-1]= [10*X/a_{\overline{10|}} - X] +486.091$
X[0.877137 – 1.391047 + 1] = 486.091
X = 486.091/.48609 = 1000
X= 1000

[14]

5 i) **Force of Interest:-**
The force of interest can be defined as the nominal rate of interest per unit time at time t convertible momently. i.e., for each value of *t* there is a number δ (*t*) such that
$\lim_{h\to 0+} i_h(t) = \delta(t)$, where δ(*t*) is called the force of interest per unit time at time *t*.

We may also define δ (*t*) directly in terms of the accumulation factor as
$$\delta(t) = \lim_{h\to 0+} \left(\frac{A(t,t\,h) - 1}{h} \right)$$

ii) Given $i^{(2)}$=0.0775

$i=(1+ i^{(2)}/2)$^2-1 = (1+.0775/2)^2-1=1.079002-1=7.9002%

δ = ln(1+i)=ln(1.079)=7.6036%

$d^{(12)} = 12*(1-v^{(1/12)}) = 12*(1-(1/1.079)^{(1/12)}) = 7.5796\%$

$i^{(1/2)} = (1/2)*((1+i)^2-1) = 0.5*(1.079^2-1) = 8.2122\%$

iii) $(1 + 1/\ddot{a}_{\overline{t}|} - d) = \dfrac{d}{(1-v^t)} + v = \dfrac{(d+v-v^{t+1})}{(1-v^t)} = \dfrac{1-v^{t+1}}{a_{\overline{t}|}} = \dfrac{a_{\overline{t+1}|}}{}$

$\text{LHS} = \sum_1^{30} \log_{10}(1/\ddot{a}_{\overline{t}|} - d + 1) = \sum_1^{30} \log_{10}(a_{\overline{t+1}|} / a_{\overline{t}|}) = \sum_1^{30} (\log_{10} a_{\overline{t+1}|} - \log_{10} a_{\overline{t}|})$

$= \log_{10} a_{\overline{31}|} - \log_{10} a_{\overline{1}|} = \log_{10} (a_{\overline{31}|} / a_{\overline{1}|}) = \log_{10} \ddot{\mathbf{a}}_{\overline{31}|} = 1.169228$

⇨ $\ddot{\mathbf{a}}_{\overline{31}|} = 10^{1.169228} = 14.764815$

From Tables, At 6% $a_{\overline{31}|} = 13.9291$

Hence, $\ddot{a}_{\overline{31}|} = 1.06*13.9291 = 14.764846$

Hence i=6%

iv) From first principles,

$(I\ddot{a})_{\overline{n}|} = 1+2v+3v^2+4v^3+5v^4+\dots + nv^{n-1}$ @ i

Multiplying by v on both sides, $v(I\ddot{a})_{\overline{n}|} = v+2v^2+3v^3+4v^4+5v^5+\dots + nv^n$

Subtracting, $(1-v)(I\ddot{a})_{\overline{n}|} = 1+v+v^2+v^3+v^4+v^5+\dots + v^{n-1} - nv^n$

⇨ $(1-v)(I\ddot{a})_{\overline{n}|} = (1- v^n)/(1-v) - nv^n$ (Sum of a GP formula)

⇨ $d(I\ddot{a})_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n$

⇨ $(I\ddot{a})_{\overline{n}|} = \dfrac{\ddot{\mathbf{a}}_{\overline{n}|} - nv^n}{d}$ Hence Proved.

v) If i is the money rate of interest and e is the inflation rate then,
Real rate of interest i' = (i-e)/(1+e) = (6.3-1.2)/1.012 = 5.039526%

Lumpsum = PV of payments received
= v^6 @ 6.3% * 50000 * $a'_{\overline{12}|}$ @ 5.039526%
= 0.693107 * 50000 * 8.843527 = 306475.5
= 306476

vi) Accumulated fund after 10 years = 3,00,000 S_{10}

Given that : $(1+i_t) \sim \log N(\mu, \sigma^2)$

⇨ $S_{10} \sim \log N(10\mu, 10\sigma^2)$

⇨ $\log S_{10} \sim N(10\mu, 10\sigma^2)$

$\Rightarrow$ $(\log S_{10} - 10\mu)/(10\sigma^2) \sim N(0,1)$

So, the probability that the amount will become at least Rs. 5,00,000 is:

$$P(3{,}00{,}000\, S_{10} \geq 5{,}00{,}000) = 1 - P(S_{10} \leq 5/3) = 1 - P(S_{10} \leq 1.6667)$$
$$= 1 - \Phi\left(\frac{\log 1.667 - 10\mu}{\sigma\sqrt{10}}\right)$$
$$= 1 - \Phi(-0.1879)$$
$$= \Phi(0.1879)$$
$$= 0.575$$

[19]

6 (i) Option 1 : If Rs.100 is invested in bank then maturity value = 100*1.08^10
Hence IRR= 8%

Option 2 :
Purchase price per Rs.100 nominal $= 6[a_{\overline{2}|\,.07} + (v^2 a_{\overline{2}|})_{.08} + (v^4 a_{\overline{2}|\,.09}) + (v^6 a_{\overline{2}|\,.10}) + (v^8 a_{\overline{2}|\,.11})] + 110\, v^{10}{}_{0.11}$

$= 6[1.8080 + 1.5289 + 1.2462 + 0.9797 + 0.7431] + 38.74029$
$= 6*6.3059 + 38.7403 = 76.5754$

IRR is given by the equation:

$-76.5754 + 6a_{\overline{10}|} + 110\, v^{10} = 0$
At 11% LHS = -2.49973
At 10% LHS = 2.701749
By interpolation, IRR = 10.519%

Option 3:
Redemption proceeds of 105 at time 5 is reinvested at 10% for a further period of 5 years.

So, IRR is given by the equation:

$90 = 5a_{\overline{5}|} + 105\,(1.1)^5\, v^{10}$
At 9% RHS = 90.87942
At 10% RHS = 84.15067
By interpolation, IRR = 9.125%

Since the IRR obtained is maximum under option 2, the investor should select Option 2.

(ii) The other criteria that can be used to decide between alternative investment projects are: (Any two)

1. Net present value and accumulated profit
2. Payback period
3. Discounted payback period

[11]

7 a) Money weighted rate of return is given by the equation:
$8.6(1+i)^2 + 0.6(1+i)^{7/4} - 1.0(1+i) = 12.0095$
For i = 20%, LHS = 12.0095
So, **MWRR = 20% p.a.**

b) Time weighted rate of return is given by:
$$(1+i)^2 = \frac{8.4}{8.6} \times \frac{(10.5+1.0)}{(8.4+0.6)} \times \frac{12.0095}{10.5}$$

$(1+i)^2 = 1.427486$
$\Rightarrow$ $i = \sqrt{1.427486} = 0.194774$
$\Rightarrow$ **TWRR = 19.48% p.a.**

c) The effective rate of return for year 2008 is given by solving the equation of value:
$8.6(1+i) + 0.6(1+i)^{3/4} = 11.5$
$\Rightarrow$ $i = 0.2545$

The effective rate of return for 2009 is given by:
$$(1+i) = \frac{12.0095}{10.5} = 1.14376$$

So, the linked annual rate of return is given by:
$(1+i)^2 = 1.2545 \times 1.14376$
$\Rightarrow$ $i = 0.1979$
$\Rightarrow$ **LIRR = 19.79% p.a.**

d) (i) The money weighted rate of return will increase as the new money received (0.6) which will accumulate for a smaller period but gives the same Fund Value (12.0095) at the end.
(ii) The time weighted rate of return will decrease as the second growth factor in the equation given in b) above will decrease.

[10]

8. (i) Any four of the following :

- Corporate Bonds are part of loan capital of companies.
- They are more risky (less secure) than the Government bonds
- They are usually less marketable than Government Bonds
- The lower security and marketability means that investors require a yield greater than on the corresponding government bonds.
- The investors lend a lump sum of money to the company. In return the company pays regular interest payments and a final payment representing a return of capital at the end of the term of the contract.
- The level of security depends on the type of bond, the company which has issued it, whether they are secured on some assets of the issuing companies and the term.

(ii) We first check if there is capital gain on redemption.

$g(1-t_1) = 0.10 \times 0.8 = 0.08$

$i^{(2)}{}_{0.08} = 0.078461$

As $g(1-t_1) > i^{(2)}$, there is capital loss.

The stock is redeemable at the option of the investor. The investor will wish to defer the capital loss as long as possible, so we assume that the investor will choose the latest possible redemption date and the bond will be redeemed after 20 years.

Let P be the maximum price that Investor X can pay in order to achieve a net yield of 8% p.a.
$P = 0.8 \times 10\, a^{(2)}_{\overline{20}|} + 100\, v^{20}$ @ 8%
$P = 8 \times 9.8181 \times 1.019615 + 100 \times 0.21455$
P = Rs. 101.54

iii) Again to check if there is any capital gain
$g(1-t_1) = 0.10 \times 0.75 = 0.075$

$i^{(2)}{}_{0.09} = 0.088061$

As $g(1-t_1) < i^{(2)}$, there is capital gain.

This means Investor Y will want to make the capital gain as soon as possible, so we assume that the bond will be redeemed at the earliest possible date i.e. after 12 years and 10 months.

Let P_1 be the maximum price that Investor Y can pay in order to achieve a net yield of 9% p.a. Equation of value is
$P_1 = (1.09)^{2/12} [0.75 \times 10\, a^{(2)}_{\overline{13}|} + 100\, v^{13} – 0.3(100-P_1)v^{13}]$ @ 9%
$P_1 = (1.0144666)x[7.5x7.4869x1.022015 + 0.7x100x0.32618 +0.3P_1x0.32618$
P_1 = Rs. 90.35

iv) Actual net yield, i, obtained by Investor X is given by solving the equation

$101.54 = 0.8 \times 10\, a^{(2)}_{\overline{2}|} + 90.35\, v^{26/12}$ @ i%
At 3% RHS = 100.1667
At 2% RHS = 102.1652
By interpolation, i = 2.3097%

[16]
[Total 100]

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