Institute of Actuaries of India

Subject ST6 – Finance & Investment B

May 2008 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping Candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1. a. The delta of the collar is calculated as follows:

Position	Delta
Buy stock	1.0
Buy put, K = Rs. 950	$\Phi(d_1) - 1 = -0.45$
Write call, K = Rs. 1050	$-\Phi(d_1) = -0.30$
Total	0.25

If the stock price decreases by Rs. 1, then the value of the collar decreases by 25 paise. The stock will be worth Rs. 1 less, the gain on the purchased put will be 45 paise, and the call written represents a *liability* that decreases by 30 paise.

b. If S becomes very large, then the delta of the collar approaches zero. Both $\Phi(d_1)$ terms approach 1. Intuitively, for very large stock prices, the value of the portfolio is simply the (present value of the) exercise price of the call, and is unaffected by small changes in the stock price.

As S approaches zero, the delta also approaches zero: both $\Phi(d_1)$ terms approach 0. For very small stock prices, the value of the portfolio is simply the (present value of the) exercise price of the put, and is unaffected by small changes in the stock price.

[6]

2. a. From parity: $F_0 = 5{,}500e^{.09/4} - 40 = 5585.15$

Actual F_0 is 5580; so the futures price is 5.15 below the "proper" level.

b. Buy the relatively cheap futures, sell the relatively expensive stock and lend the proceeds of the short sale:

	CF Now	CF in 6 months
Buy futures	0	$S_T - 5580$
Sell shares	5500	$-S_T - 40$
Lend Rs. 5500	–5500	5625.15
Total	0	5.15

c. If we call the original futures price F_0, then the proceeds from the long-futures, short-stock strategy are:

	CF Now	CF in 6 months
Buy futures	-10	$S_T - F_0$
Sell shares	5450	$-S_T - 40 - 50$
Place Rs. 5450 – 10 in margin account	–5440	+5440
Total	0	$5350 - F_0$

Therefore, F_0 can be as low as 5350 without giving rise to an arbitrage opportunity. {It is assumed that the broker will allow the brokerage on stocks and futures to be adjusted with the margin account. If not then this amount has to be borrowed and repaid later incurring a loss of interest on this amount which would push down the cash flow in six months time. It is likely that the broker may allow the brokerage on stocks to be adjusted with the margin but not on the futures in which case it will only be the interest on Rs 10. All the three answers would be acceptable. }

On the other hand, if F_0 is higher than the parity value (5585.15), then there is no short-selling cost but only brokerage cost would exist. To do arbitrage, one would buy stock and sell futures and reverse this transaction later. Twice the brokerage cost on stocks and once the brokerage cost on futures would be incurred which implies that there can be no arbitrage as long as the futures price is not higher than 5585.15 + 2*50+40 i.e. 5735.15 Therefore, the no-arbitrage range is:

$$5350 \le F_0 \le 5735.15$$

d.
1) Zero bid offer spread
2) Zero impact cost
3) Large/fractional supply of stocks at a given price
4) Zero tax
5) Zero credit/counterparty risk
6) Zero operational risk like settlement/theft/fraud etc.
7) Borrowing possible at risk free rate (No credit risk for yourself)
8) Short selling allowed without settlement for 6 months

[10]

3(a).
$$df = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} dF + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}(dF)^2$$

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} F\mu + \frac{1}{2}\frac{\partial^2 f}{\partial F^2} F^2\sigma^2\right)dt + \frac{\partial f}{\partial F} F\sigma dZ$$

b. Because it costs nothing to enter into a futures contract, the portfolio given in the equation has value:

$V = -f$ [0.5 mark]

In a time period δt, the holder of the portfolio earns capital gain equal to δf from the derivative and income of $\frac{\partial f}{\partial F}\delta F$ from the futures contract.

Using this, the change in the value of the portfolio over a small time interval dt is

$$dV = -df + \frac{\partial f}{\partial F}dF = -df + \frac{\partial f}{\partial F}(\mu F dt + \sigma F dZ)$$

Now substituting in the expression of df from the above we get

$$dV = -\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial F}F\mu + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}F^2\sigma^2\right)dt - \frac{\partial f}{\partial F}F\sigma dZ + \frac{\partial f}{\partial F}(\mu F dt + \sigma F dZ)$$

$$dV = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}F^2\sigma^2\right)dt$$

We notice that this change in value depends only on the change in time, not on the random increment dZ. The portfolio is therefore risk-free and by the no-arbitrage principle, we must have

$$dV = rVdt$$

$$dV = -rfdt$$

Equating our two expressions for dV gives

$$dV = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}F^2\sigma^2\right)dt = -rfdt$$

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}F^2\sigma^2\right) = rf$$

[4]

4 a). The price of the bond and of an equivalent bond, denoted by P_{cb} and P_{gb} respectively will be:

$P_{cb} = 10e^{-0.08} + 110e^{-0.08x2} = 102.967$

$P_{gb} = 10e^{-0.06} + 110e^{-0.06x2} = 106.979$

The difference in price is 106.979 – 102.967 = 4.012. This represent the present value of the amount expected to be lost through default over the lifetime of the bond.

Time	Default Probability	Recovery Amount	Risk-free value	Conditional Loss	Expected PV of Loss
1	Q	30	$10+110e^{-0.06}$ = 113.594	83.594	$83.594e^{-0.06}$ Q= 78.726Q
2	Q	30	110	80	$80e^{-0.06x2}$ Q= 70.954Q
Total					149.68Q

The total expected present value of default losses is 149.68Q. Combining this with our earlier reasoning gives:

149.68Q = 4.012

Q = 0.0268 = 2.68%

4 b). CDS is the most popular type of credit derivative. This is a contract that provides insurance against the risk of a default on a bond issued by a particular company. The bond is known as the reference entity and a default by the company is known as an event. The buyer of the insurance obtains the right to sell bonds issued by the company for their face value when such an event occurs. The buyer of the CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occur.

The settlement can be done either in physical or in cash. In the physical settlement the buyer of protection sell bonds issued by the reference entity for their face value. In case of cash settlement, a calculation agent estimates the value of the cheapest-to-deliver bond issued by the company a specified number of days after the default event. The cash payoff is then based on the excess of the face value of these bonds over estimated value.

[8]

5. a) In Monte Carlo Simulation sample values for the derivative security in a risk neutral world are obtained by simulating paths for the underlying variable. The total time is divided into a discrete set of time points. At each such time point t_i the increment in the Brownian motion is generated from a standard normal distribution and hence a value of the stock price is generated at this time by using the discretized version of the differential equation. Hence, we can generate the stock prices at all the discrete time points which gives one sampled path. For each path, the value of the asian option is known at maturity and it can be discounted to time 0 to get the corresponding sample value of derivative. Since the paths are sampled in the risk neutral world the average of the sample values obtained would be an estimate of the value of the derivative. This estimate tends to the accurate value of the derivative as the number of sampled path increases.

On each simulation run only one sample path is obtained and this path is simulated in the forward direction, values for the underlying variables are first determined at time Δt, then at time $2\Delta t$, then at time $3\Delta t$, etc. At time $i\Delta t$ ($i = 0,1,2,\ldots$) it is not possible to determine whether early exercise is optimal since the range of paths which might occur after time $i\Delta t$ have not been investigated. Since the simulation is accounting for only one path at a time, it is difficult to say whether the optimal stopping time has reached or not because to know when to stop we should account for all possible paths in the future and not just the sampled path. Other numerical procedures which accommodate early exercise work by moving backwards from time T to t and accounts for multiple possible paths.

b) The control variate technique is applicable when there are two similar derivatives, A and B. Derivative A is the security under consideration; derivative B is similar to derivative A and has an analytic solution available. Numerical methods are used to estimate both f_A and f_B and a true estimate F_B for B is obtained by using analytical formula. A better estimate of the value F_A for A is then obtained by the following formula, $F_A = f_A - f_B + F_B$.

In this case, a simulation required two sets of samples from standardized normal distributions. The first is to generate the volatility movements (volatility is stochastic) and the second is to generate the stock price movements once the volatility movements are known. Now, Monte Carlo technique can be used to get the price f_A. The control variate technique involves carrying out a second simulation on the assumption that the volatility is constant. The same random number stream is used (the second one in this case) to generate the stock price movements as in the case of f_A. Then a better estimate is given by $f_A - f_B + F_B$ where f_A is the option value from monte carlo when volatility is stochastic, f_B is the option value using monte carlo when the volatility is constant and F_B is the true Black Scholes value when the volatility is constant.

[10]

6 a). In Vasicek's model and Hull-white model, the standard deviation stays at 2%. In the Cox, Ingersoll and Ross model the standard deviation of the short rate is proportional to the square root of the short rate. When the short rate increases from 9% to 16% the standard deviation of the short rate increases from 2% to 8/3%. In the Rendleman and Bartter model the standard deviation is proportional to the level of the short rate. When the short rate increase from 9% to 16% the standard deviation increases from 2% to 32/9%.

6 b) The friend trades in derivative and hence it is his job to create a replicating portfolio for the derivative. Theory says that such a portfolio would require continuous rebalancing which in practice would be in-frequent rebalancing due to transaction costs. It is advantageous if such a portfolio is equally sensitive to various factors as is the option price and hence would require less rebalancing. So, it will be advantageous that sensitivity of this portfolio equals the sensitivity of the option price. The most important sensitivities with regard to this are delta and gamma. In case of stocks, delta and gamma risks are associated with price of underlying asset which is easily observable and is a single dimension except that volatility is

not observable and needed an estimate. But in case of caps and floors, these risks are associated with a shift in the zero curve, which is not observable along with volatility. A number of alternative deltas can be defined since the curve can shift in a multiple ways. When several delta measures are calculated, there are many possible gamma measures. The problem is greater compared to the stocks because in this case both the time zero value and volatility is not observable. Besides the time zero value is not a single dimension. This is certainly difficult in this case compared to the case of stocks.

6 c) Numerical values must be assigned to the constants α and σ. These can be obtained from historical data. If we need to model the short rate up to time T, we need an estimate of $\mu(t)$ for times $t <= T$. Obtain the set of observed zero coupon bond prices at time zero. Fit a differentiable interpolating function g(0,t) that matches the observed prices at each of the relevant term. Use the equation $-\frac{\partial}{\partial t} g(0,t)$ to find the current curve of forward rates. Then the function $\mu(t)$ can be estimated using a formula based on the observed forward rate curve and the estimates of α and σ.

[12]

7(a) The fund manager should short

$$\frac{(0.8-0.5)x800{,}000{,}000}{5400x50} = 888.89$$

Around 889 contracts should be shorted.

7 b) There can be various reasons for the same. The beta measured by the manager is the realized beta and the realized beta will be different than the prospective beta. It also depends on the error in the estimated beta at the start (0.8). It is assumed that the beta of the futures would be equal to 1 exactly but it may be different because of the stochastic basis. It is assumed that there is no change in the beta over this period but it may change either due to trading or due to change in weights which may have changed through the gains/losses. The

fund manager may have to fund the gains and the losses on the futures contract utilizing the cash position rather than the stocks and certainly that would have created additional gain/losses which may not have been accounted through the stock portfolio.

[5]

8 a) There are 180 days between October 3 and March 31 and 183 days between October 3 to April 3. The cash price of the bond is, therefore

$$120+\frac{180}{183}x6 = 125.902$$

A coupon of 6 will be received in 3 days (=0.00822 years) time. The present value of the coupon is

$$6e^{-0.10x0.00822} = 5.995$$

The futures contract lasts for 61 days (=0.1671 years). The cash futures price if the contract written on the 12% bond would be

$$(125.902-5.995)e^{0.10x0.1671} = 121.9273$$

At delivery there are 58 days of accrued interest. The quoted futures price if the contract were written on the 12% bond would therefore be:

$$121.9273 - 6x\frac{58}{183} = =120.0257$$

Taking the conversion factor into account the quoted futures price is:

$$\frac{120.0257}{1.4269} = 84.1164$$

b) The cheapest-to-deliver bond is the one for which

Quoted Price – Futures Price x Conversion Factor

is least. Calculating this factor for each of the 5 bonds we get

Bond A : 130.25 – 104.50x1.3241 = -8.118

Bond B : 147.50 – 104.50x1.4753 = -6.669

Bond C : 119.25 – 104.50x1.1326 = 0.893

Bond D : 149.50 – 104.50x1.4962 = -6.853

Bond E : 120.25 – 104.50x1.1563 = -0.583

Bond A is the cheapest to deliver.

[8]

9)

i) Longevity bonds

ii) Mortality Swaps

iii) Survivor caps / caplets

iv) Principal at risk longevity bonds

Any of the 3 from the following:

i) A longevity bond is a traded security and is a form of index linked, amortization bond, that pays an amount k x S(t) per unit at time t for t = 1, 2, … T, where T is the maturity of the bond, k is the nominal coupon rate, S(t) = p(0, x) x ….. x p(t-1, x) is called the survivor index and p(t, x) is the measured probability (retrospectively) that an individual aged x at time 0 survives from t to t+1. Since the payout depends on the survival probability with reference to an indexed population, this can be used to hedge the probability of survival if the policy holder population is similar to the indexed population. If they are different then there would be a basis risk. Since annuity payouts are linked to the probability of survival, a pension plan can simulate these by buying an appropriate number of longevity bonds.

ii) A mortality swap is an OTC contract, that swaps the floating amount kS(t) for a fixed amount $kS_1(t)$ for t = 1, 2, … T, where $S_1(1)$, ……. , $S_1(T)$ are fixed at time 0 when the OTC contract is arranged. [1.5 marks] If the pension plan enters into such an OTC contract where S_1 represents its assumed probability of survival and the index S represents the actual behavior then the longevity risk will be perfectly hedged. As argued in part(i) it is unlikely that the index will be defined on a different population compared to the insurers and this will then give rise to basis risks. Please note that there will be reduced basis risk because the counterparty would be happy to take the closest standard population as reference/index.

iii) Survivor caps and caplets are OTC contracts and derive their names from interest-rate caps and caplets. A survivor caplet will pay, at time t, the maximum of $S(t) - S_1(t)$ and 0. Here, $S(t)$ and $S_1(t)$ are as defined above in (ii). A survivor cap is simply a collection of survivor caplets with payment dates t = 1, 2, … T just like the relationship between cap and caplet. The concept here is that a pension plan might be concerned specifically about high survival rates and hedging against this excess risk while continuing to take surplus if survival rates are lower than anticipated. A survivor caplet/cap can provide the right sort of payoff under these circumstances.

iv) A principal-at-risk longevity bond shares characteristics with a corporate bond and may be a fixed or floating rate bond. These are generally traded securities. The bond pays coupons at a specified rate that are not affected by changes in aggregate mortality rates followed by a repayment of principal that may be reduced if aggregate mortality rates are different from anticipated. In this sense the reduction in principal is similar to default on a corporate bond at the maturity date. A simple example might be that the principal repaid at T is equal to $100 - 100 \max(S(T) - S_1(T), 0)$: that is it is reduced if the survivor index at T is above a certain threshold. Please note this simple structure will only allow to hedge the longevity risk at time T. One can link the repayment of principal to each of S(1), … ,S(T) making it a form of path dependent option. For this to be a good hedge, such a function has to be carefully chosen to make it effective hedge.

[10]

10)

Portfolio insurance means that the portfolio does not fall below the floor because if it falls somebody will pay. Portfolio managers sometimes issue scheme guaranteeing a floor value for the portfolio and then to hedge the guarantee risk they need to buy a put option. Since derivative markets may not be deep, the fund manager may synthesize the option contract by regularly buying the stocks and lending/borrowing money. At times the manager may use futures in place of stocks because futures contract are easier to handle. The option theory tells us that when the market goes down, the replicating portfolio for put will keep on shorting the stock to match the gains on the put contract and when the market goes up it will keep buying the stocks. Therefore, when the market declines, they cause portfolio managers either to sell stocks or to sell index future contracts. Either action may accentuate the decline. The sale of stock is liable to drive down the market index further in a direct way. The sale of index futures contract is liable to drive down futures prices. This creates selling pressure on stocks via the mechanism of index/futures arbitrage so that the market index is liable to be driven down in this case as well. Therefore, if everyone follows these strategies then they will not work because prices would keep falling and there will be no buyers. Similarly, when the market rises, the portfolio insurance schemes cause portfolio managers either to buy stock or to buy futures contracts. This may accentuate the rise.

[6]

11) a) Average price asian put option

b) Lookback call option

c) Down and out put option

d) Asset or nothing binary call option

e) $\text{Abs}(S_T - K) = \text{Max}(\text{Max}(S_T - K,0)\ ,\ \text{Max}(K - S_T\ ,\ 0)) = \text{Max}(c_T, p_T)$. i.e. simple chooser option (as you like it).

f) Exchange option where option to receive one unit of V in exchange for one unit of U.

[6]

12)

a) The payoff is equal to Max($S_2 - S_1$,0), based on the final share prices. If the share prices are strongly positively correlated, S_2 and S_1 will tend to move in the same direction, so that $S_2 - S_1$ will vary over a fairly narrow range and there is little chance of a large payoff. If the share prices are negatively correlated, S_2 and S_1 will tend to move in the opposite direction, so that $S_2 - S_1$ will vary over a much wider range and there is the possibility of a large payoff. So we would expect the price of the option to be higher when the correlation is negative and highest when $\rho = -1$.

b) We can calculate the fair price of the opposite exchange option by interchanging the 1's and 2's in the formula. Hence,

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sqrt{0.21^2 + 0.3^2 - 2(0.5)(0.3)(0.21)} = 0.267$$

$$d_1 = \frac{\log\left(\frac{S_1}{S_2}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{420}{400}\right) + \frac{1}{2}0.267^2(0.5)}{0.267\sqrt{0.5}} = 0.352$$

$d_2 = d_1 - \sigma\sqrt{T} = 0.352 - 0.267\sqrt{0.5} = 0.162$ (0.25 for sigma, 0.25 for d's)

value of opposite exchange option $f_{opp} = 420\Phi(0.352) - 400\Phi(0.162) = 41.82$. (0.25)

c) We would expect the following put-call parity relationship to apply

$$f + S_1 = f_{opp} + S_2$$

This is because, if we set up portfolios corresponding to each side of the equation now, both would be worth max(S_1, S_2), based on the final share prices, at the maturity date. So, by the principle of no-arbitrage, the portfolios should have equal values now.

So, the price in part b should be $f_{opp} = f + S_1 - S_2 = 22 + 420 - 400 = 42$. This is slightly different than the calculated value in part b which may either be due to wrong inputs or due to approximations involved in the calculation.

[8]

13)

a) Solution To calculate $E(M_{n+1}|F_n)$, first note that M_{n+1} is equal to $(A_n+1)/(A_n+B_n+1)$ with probability $M_n = A_n/(A_n+B_n)$ and M_{n+1} equals $A_n/(A_n+B_n+1)$ with probability 1-M_n. Thus,

$$E(M_{n+1}|F_n) = \frac{A_n+1}{A_n+B_n+1}\frac{A_n}{A_n+B_n} + \frac{A_n}{A_n+B_n+1}\frac{B_n}{A_n+B_n}$$

$$= \frac{1}{A_n+B_n+1}\left\{\frac{A_n(A_n+B_n+1)}{A_n+B_n}\right\} = \frac{A_n}{A_n+B_n} = M_n$$

b) Since both started with 1 mn, therefore to begin with the total amount is 2 and with every customer they got 1 mn implying for n customers they got n millions (which is divided among them). Since investment returns is zero (also assumed is expenses is zero) $A_n + B_n = n+2$ and $M_n = A_n/(n+2)$

c) For n=1, $P(A_1=1) = P(A_1=2) = 1/2 = 1/(1+1)$ i.e. the equation holds for n=1. To use induction, assume that it holds for n-1. Now we will prove it for n.

For A_n to be equal to k, A_{n-1} should be either k with the n[th] customer giving the money to Bob or k-1 with n[th] customer giving the money to Alice. Probability that the n[th] customer will give the money to Alice given that Alice has k amount is equal to k/(n-1+2).

Hence,

$P(A_n = k) = P(A_{n-1} = k) * (1 - k/(n+1)) + P(A_{n-1} = k-1) * (k-1)/(n+1)$.

Substituting, the general formula $P(A_n = k) = 1/(n+1)$ in the above equation we get,

$$P(A_n = k) = \frac{1}{n}\frac{(n+1-k)}{n+1} + \frac{1}{n}\frac{(k-1)}{n+1} = \frac{1}{n+1}$$

We have shown that the above equation holds for n=1 and also that it holds for any n>1 given that it holds for n-1. Since it holds for 1, hence it should hold for 2 and by induction for all finite n.

[8]

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