Institute of Actuaries of India

Subject CT8-Financial Economics

May 2008 Examination

INDICATIVE SOLUTION

Q.1

(a)

$F(0,5,6) = (1/(6-5))*(\ln((B(0,5)/B(0,6)))$

Where,

$F(0,5,6)$ is forward rate at time 0 for delivery between time 5 and 6
$B(0,5)$ is zero coupon bond price at time 0 for Re 1 payable at time 5
$B(0,6)$ is zero coupon bond price at time 0 for Re 1 payable at time 6

Therefore, $F(0,5,6) = (1/1)* \ln(0.72/0.68) = 5.716\%$

(b) Market price of risk:

Market price of the risk represents the excess expected return over the risk-free rate per unit of volatility in return for an investor taking on this volatility.

$\gamma(t,T) = (m(t,T) - r(t))/S(t,T)$

(c) One-factor model:

A one-factor is one in which interest rates are assumed to be influenced by a single source of randomness

The prices of all bonds (of all maturities) and interest rate derivatives must therefore move together.

The randomness is usually modeled as an Itó process.

The stochastic differential equation for r(t) has the following form under the real-world probability measure P:

$dr(t) = a(t,r(t))dt + b(t,r(t))dW(t)$

Where a(.) and b(.) are appropriately chosen functions

Limitations:

The prices of bonds of different terms in the real world are not observed to be perfectly correlated, always moving together.

Sustained periods have been observed historically with all combinations of high/low interest rates and high/low volatility. This seems to be inconsistent with one-factor model.

Some interest rate products are explicitly dependent on other variables, which would be expected to introduce a separate source of randomness.

Total Marks: 9

Q.2

(a)

By put-call parity we know that
$c_t + k * \exp(-r * (T-t)) = p_t + s_t$

therefore, $p_t = 187.06 + 5{,}250 * \exp(-0.05\times0.5) - 5{,}000$
$= 307.44$

(b)

Implied volatility:

$c_t = St * \Phi(d_1) - K*(\exp(-r * (T-t))) * \Phi(d_2)$

where $d_1 = \{\log(St/K) + (r + (\frac{1}{2}\sigma^2)) * (T- t)\} /(\sigma * \sqrt{(T-t)})$

$d_2 = d_1 - (\sigma * \sqrt{(T-t)})$

if $\sigma = 0.15$, $\mathbf{c_t = 158.96}$

as $d_1 = -0.1713$ and so $\Phi(d_1) = 0.4320$
and $d_2 = -0.2773$ and so $\Phi(d_2) = 0.3908$

if $\sigma = 0.18$, $\mathbf{c_t = 200.72}$

as $d_1 = -0.1233$ and so $\Phi(d_1) = 0.4509$
and $d_2 = -0.2506$ and so $\Phi(d_2) = 0.4011$

$k = 5{,}250$
$s_t = 5{,}000$
$r = 0.05$
$k * \exp(-r * (T-t)) = 5120.3769$

By linear interpolation, $\sigma = 0.17$
at $\sigma = 0.17$, $c_t = 187.06$,

$d_1 = 0.1378$ and so $\Phi(d_1) = 0.4452$
and $d_2 = 0.2580$ and so $\Phi(d_2) = 0.3982$

can therefore price a call with strike of 4,750 and volatility = 0.17 using formula in (b)

c_t (k = 4,750) = 459.38

as $d_1 = 0.6948$ and so $\Phi(d_1) = 0.7564$
and $d_2 = 0.5746$ and so $\Phi(d_2) = 0.7172$

p_t (k = 4,750) = 92.10

k = 4,750
s_t = 5,000
r = 0.05
k * exp(−r * (T−t)) = 4632.722

Total Marks: 10

Q.3

(1)(a)

One-step transition probability matrix is:

$$\mathbf{P} = \begin{array}{c} \\ \text{State1} \\ \text{State2} \\ \text{State3} \\ \text{State4} \end{array} \begin{array}{c} \begin{array}{cccc} \text{State1} & \text{State2} & \text{State3} & \text{State4} \end{array} \\ \begin{bmatrix} 0.650 & 0.200 & 0.100 & 0.050 \\ 0.150 & 0.600 & 0.200 & 0.050 \\ 0.000 & 0.150 & 0.800 & 0.050 \\ 0.000 & 0.000 & 0.150 & 0.850 \end{bmatrix} \end{array}$$

$$\mathbf{P}^2 = \begin{bmatrix} 0.650 & 0.200 & 0.100 & 0.050 \\ 0.150 & 0.600 & 0.200 & 0.050 \\ 0.000 & 0.150 & 0.800 & 0.050 \\ 0.000 & 0.000 & 0.150 & 0.850 \end{bmatrix} \begin{bmatrix} 0.650 & 0.200 & 0.100 & 0.050 \\ 0.150 & 0.600 & 0.200 & 0.050 \\ 0.000 & 0.150 & 0.800 & 0.050 \\ 0.000 & 0.000 & 0.150 & 0.850 \end{bmatrix}$$

$$= \begin{bmatrix} 0.453 & 0.265 & 0.193 & 0.090 \\ 0.188 & 0.420 & 0.303 & 0.090 \\ 0.023 & 0.210 & 0.678 & 0.090 \\ 0.000 & 0.023 & 0.248 & 0.730 \end{bmatrix}$$

$$P^3 \begin{bmatrix} 0.650 & 0.200 & 0.100 & 0.050 \\ 0.150 & 0.600 & 0.200 & 0.050 \\ 0.000 & 0.150 & 0.800 & 0.050 \\ 0.000 & 0.000 & 0.150 & 0.850 \end{bmatrix} \begin{bmatrix} 0.453 & 0.265 & 0.193 & 0.090 \\ 0.188 & 0.420 & 0.303 & 0.090 \\ 0.023 & 0.210 & 0.678 & 0.090 \\ 0.000 & 0.023 & 0.248 & 0.730 \end{bmatrix}$$

$$= \begin{bmatrix} 0.334 & 0.278 & 0.266 & 0.122 \\ 0.185 & 0.335 & 0.358 & 0.122 \\ 0.046 & 0.232 & 0.600 & 0.122 \\ 0.003 & 0.051 & 0.312 & 0.634 \end{bmatrix}$$

Using the probabilities calculated above, the risk-neutral expected amounts of the payments are:

Time 1: 6.5*0.65 + 6.5*0.75*0.2+ 6.5*0.5*0.1 = 5.525

Time 2: 6.5*0.453 + 6.5*0.75*0.265+ 6.5*0.5*0.193 = 4.85875

Time 2: 116.5*0.334 + 116.5*0.75*0.278+ 116.5*0.5*0.266 = 78.70

Present values of these risk-neutral expected amounts:

5.525/(1.0625)^1 + 4.485875/(1.0625)^2 + 78.70/(1.0625)^3 = 75.116

(1)(b)

The risk-neutral present value calculated above is the fair value of the bonds.

Therefore, if the financial institution buys the bonds at par (Rs. 100), it will be paying too much of the price.

(2)

The rate of return the financial institution will earn if the bond issuer makes payments in full.

6.5/(1+i)^1 + 6.5/(1+i)^2 + 116.5/(1+i)^3 = 95.7

By trial and improvement, i = 11.25%

The credit spread is the difference between this and the yield on corresponding default-free bond.

So, credit spread = 11.25 – 6.25 = 5%

Total Marks: 10

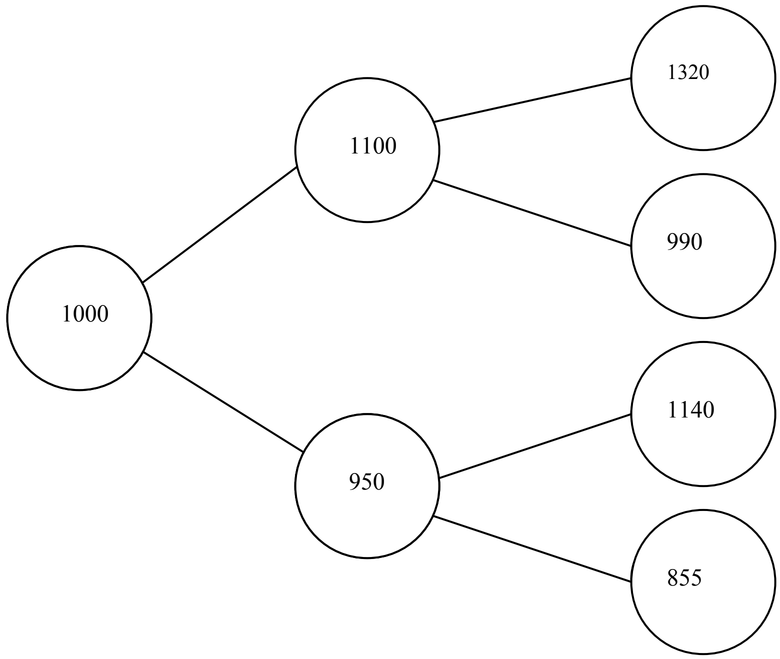
Q.4 A recombining binominal tree or binominal lattice is one in which the sizes of the up-steps and down-steps are assumed to be the same under all states and across all time intervals.
i.e., $u_t(j)=u$ and $d_t(j)=d$ for all times t and states j, with $d < \exp(r) < u$

• It therefore follows that the risk neutral probability 'q' is also constant at all times and in all states eg. $q_t(j)=q$

• The main advantage of a 'n' period recombining binominal tree is that it has only [n+1] possible states of time as opposed to 2^n possible states in a similar non-recombining binominal tree. This greatly reduces the amount of computation time required when using a binominal tree model.

• The main dis-advantage is that the recombining binominal tree implicitly assumes that the volatility and drift parameters of the underlying asset price are constant over time, which assumption is contradicted by empirical evidence.

b) i.



The risk-neutral probabilities at the first and second steps are as follows:

q_1= (exp(0.0175) - 0.95)/(1.10-0.95)

= (0.06765)/0.15
= 0.4510

q_2 = (exp(0.025) - 0.90)/(1.20-0.90)

= 0.41772

Put payoffs at the expiration date at each of the four possible states of expiry are 0,0,0 and 95.

Working backwards, the value of the option V_1 (1) following an up step over the first 3 months is

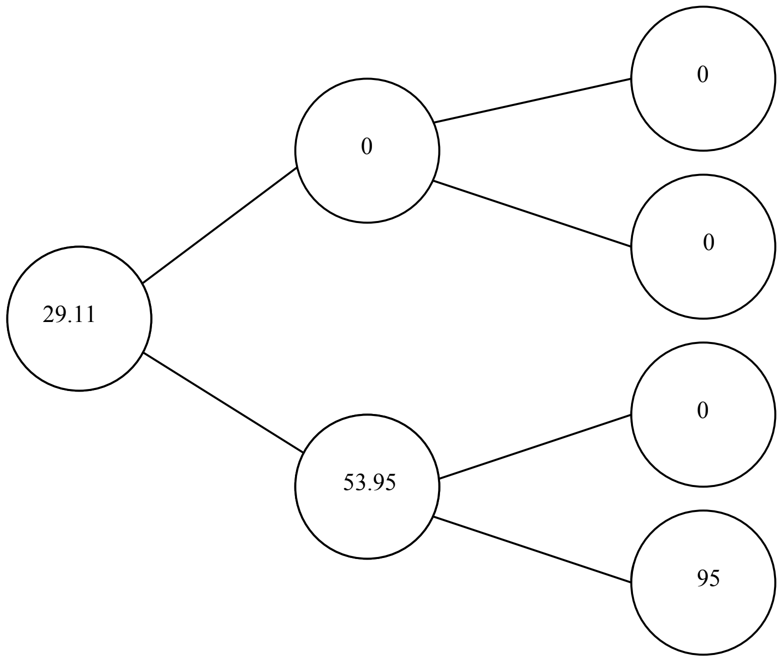
V_1 (1) exp(0.025) = [0.41772* 0] + [0.58228*0]
i.e., V_1 (1) = 0

The value of the option V_1 (2) following a down step over the first 3 months is:

V_1 (2)exp(0.025) = [0.41772* 0] + [0.58228*95]
i.e., V_1 (2) = 53.9508

The current value of the put option is:

$V_0 \exp(0.0175) = [0.4510*0] + [0.5490*53.9508]$
i.e., $V_0 = 29.105$



b) ii.

- While the proposed modification would produce a more accurate valuation, there would be a lot more parameter values to specify. Appropriate values of u and d would be required for each branch of the tree and values of 'r' for each month would be required.

- The new tree would have 2^6 =64 nodes in the expiry column. This would render the calculations prohibitive to do normally , and would require more programming and calculation time on the computer.

- An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

Total Marks: 14

Q.5

(a)

An arbitrage opportunity is a situation where sure profit can be made with no risk.

We can start at time 0 with a portfolio that has a net value of zero.

At some future time T:

- The probability of a loss is 0
- The probability to make strictly positive profit is greater than 0.

In an efficient market it is difficult to find an arbitrage opportunity because all the active participants in the market would avail this opportunity and soon the market prices of the assets would change to remove the arbitrage opportunity.

(b)

The 1 month cost of borrowing money to buy 1 share:
exp(6/1200)*(1400.70) – 1400.70

= 1407.72 – 1400.70
= 7.02

Arbitrage opportunity:

Borrow Rs. 1400.70 for one month and buy 1 share of Infosys
Sell 1 future contract of Infosys at Rs. 1410.60.

On 31 May, 2008, sell 1 share of Infosys and buy 1 future contract of Infosys

Profit after I month will be = 1410.60 – 1407.72 = Rs. 2.88

(c)

Lower bound:

Consider a portfolio A, consisting of a European put option on a non-dividend paying share and a share.

Compare this with the alternative of cash, currently worth K*(exp(-r(T-t))), where

r	:	risk-free interest rate
T-t	:	Time to expiry
K	:	Strike price
St	:	Underlying share price at time ‘t’

At time T, portfolio A will be worth at least as much as the cash, because,

At time T, cash will be worth K.

Portfolio A (the share plus the put option) will be worth:

- K if $S_T < K$ (because the option will be exercised by selling the share, leaving K)
- S_T if $S_T > K$ (because the option will not be exercised)

Thus the portfolio A is always worth at least as much as the cash deposit at time T.

Therefore, $p_t + St \geq K^*(\exp(-r(T-t)))$

So, $p_t \geq K^*(\exp(-r(T-t))) - St$

Upper bound:

For an European put, the maximum value obtainable at expiry is the strike price K. Therefore, the current value must satisfy:

$p_t \leq K^*(\exp(-r(T-t)))$

It can't exceed the discounted value of sum received on exercise, which it will equal if the share price falls to zero.

(d)

Consider two portfolios:

Portfolio A:

An European call option plus cash worth K*(exp(-r(T-t))). The value of portfolio A at the expiry date will be:

- K if $S_T \leq K$ (because the option will not be exercised)
- S_T if $S_T > K$ (because the option will be exercised)

Portfolio B:

Underlying share plus an European put option with the same expiry date and exercise price as the call. The value of portfolio A at the expiry date will be:

- K if $S_T \leq K$ (because the option will be exercised)
- S_T if $S_T > K$ (because the option will not be exercised)

Thus, the values at expiry are the same for both portfolios regardless of the share price at that time, ie max(K, S_T).

Since they have the same value at expiry and since the options can't be exercised before then they should have the same value at any time t<T.

Therefore,

$c_t + K*(\exp(-r(T-t))) = p_t + St$

This relationship is known as put-call parity.

Total Marks: 16

Q 6 – Solution
a) Explain the different forms of efficient market hypothesis (EMH).

The different forms of EMH are
Strong Form – Market prices incorporate all information available to the public and insiders.
Semi-strong form – Market prices incorporate all publicly available information
Weak form – Market prices incorporate all information of historical prices

b) Does weak form of EMH imply that the strong form is applicable? Does the strong form of EMH imply that the weak form is applicable? Explain.

Historical data is publicly available information. Thus, the strong form implies that the weak form applies. However, data other than historical prices may impact stock prices, e.g. plans for introduction of a new product that may not be known to the public. Thus, weak form need not imply that the strong form holds.

c) How do the following relate to the EMH?
- **Technical Analysis**
- **Fundamental Analysis**
- **Insider Trading**

- Weak form of EMH implies that the study of past prices cannot be used to predict future price movements. This challenges the effectiveness of technical analysis. Similarly, if technical analysis works, it implies that the market is not even weak form efficient.

- Fundamental analysis relies on publicly available information for predicting stocks that are underpriced or overpriced. The semi-strong form of EMH thus challenges the effectiveness of fundamental analysis.

- Insider trading regulations imply that price sensitive information exists and is not reflected in the prices. This suggests that the strong form of EMH may not hold. It can also be argued that since insider trading is illegal, the information cannot be acted upon and the market is efficient because of legislation.

Total Marks: 9

7(a)

$R_P = 100x^2 - 4000$; where x is the uniform (1,10) distribution
Variance of returns = 10000var(x^2)
var(x^2) = E(x^4) – [E(x^2)]^2
E(x) = (1+10)/2 = 11/2
Var(x) = 81/12 = 27/4
E(x^2) = var(x) + [E(x)]^2 = 27/4 + 121/4 = 37
E(x^4) = $\int_1^{10} \frac{x^4}{9} dx = 99999/45 = 2222.2$
var(x^2) = 2222.2 – 1369 = 853.2
Variance of returns = 8532000
E(R_P) = 100E(x^2) – 4000 = -300

Downside semi-variance of returns = $\int_1^{\sqrt{37}} (-300 - 100x^2 + 4000)^2\, dx$

Downside semi-variance of returns = 44412277 – 13445333 = 30966943

7(b)
The value of x for P(x) < 0.1 = 1.9
Var at 10% confidence interval = 361 – 4000 = -3639

Total Marks: 11

8(a)

Let the proportion of security A and security B in the portfolio be w_A and w_B respectively.
The expected return of the portfolio is given by:
$E(R_p) = w_A \times E(R_A) + w_B \times E(R_B)$
The variance of returns of the portfolio is given by
$Var(R_p) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B Cov(R_A, R_B)$

Since the correlation is -1,
$Cov(R_A, R_B) = -1 \times \sigma_A\sigma_B$

and,

$$Var(R_p) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 - 2w_A w_B \sigma_A \sigma_B$$
$$= (w_A\sigma_A - w_B\sigma_B)^2$$

For the portfolio to be riskless,

$$Var(R_p) = (w_A\sigma_A - w_B\sigma_B)^2 = 0$$

or, $w_A\sigma_A = (1 - w_A)\sigma_B$

or, $w_A = \dfrac{\sigma_B}{\sigma_A + \sigma_B} = \dfrac{10\%}{10\% + 30\%} = 0.25$

Therefore, it is possible to construct a riskless portfolio.

8(b)

The variance of the portfolio is given by

$$Var(R_p) = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B$$
$$= w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)\rho_{AB}\sigma_A\sigma_B$$

To find the proportion of security A in the portfolio that minimizes the variance, we set $\dfrac{\delta Var(R_p)}{\delta w_A} = 0$.

or, $2w_A\sigma_A^2 + 2w_A\sigma_B^2 - 2\sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B - 4w_A\rho_{AB}\sigma_A\sigma_B = 0$

or, $w_A(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) = \sigma_B^2 - \rho_{AB}\sigma_A\sigma_B$

or, $w_A = \dfrac{(\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B)}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)}$

Therefore, if $\rho_{AB} = 0.1$,

Proportion of security A in the portfolio for minimum variance is

$$w_A = \frac{(0.01 - 0.003)}{(0.09 + 0.01 - 2 \times 0.003)} = 7.45\%$$

8(c)

The covariance between 2 securities I and j is given by

$$C_{ij} = Cov[R_i, R_j]$$
$$= Cov[\alpha_i + \beta_i R_M + \varepsilon_i, \alpha_j + \beta_j R_M + \varepsilon_j]$$

We know that α_i , β_i , α_j , β_j are constants.
Therefore,

$$C_{ij} = Cov[\beta_i R_M + \varepsilon_i, \beta_j R_M + \varepsilon_j]$$

$$= Cov[\beta_i R_M, \beta_j R_M] + Cov[\beta_i R_M, \varepsilon_j] + Cov[\beta_j R_M, \varepsilon_i] + Cov[\varepsilon_i, \varepsilon_j]$$

A single index model assumes that
$Cov[\varepsilon_i, R_M] = 0$

It also assumes that
$Cov(\varepsilon_i, \varepsilon_j) = 0$, when $i \neq j$

Therefore,
$C_{ij} = Cov[\beta_i R_M, \beta_j R_M]$
$= \beta_i \beta_j Cov[R_M, R_M]$
$= \beta_i \beta_j V_M$

Total Marks: 12

9(a)

Under CAPM, the risk of a security is measured as the variance of returns.

The risk comprises two components – diversifiable risk and non-diversifiable risk.

The diversifiable risk is the risk specific to the company or the industry and can be eliminated by suitable diversification of the portfolio.

The non-diversifiable or the systematic risk is component of the risk because of the market as a whole and is measured as volatility of returns of the security relative to the volatility of the market as a whole. It is denoted by β.

Under the CAPM, the market rewards only non-diversifiable risk and not does not reward diversifiable risk.

9(b)

Under CAPM, the market price of risk is defined as $\dfrac{E_M - r}{\sigma_M}$;

Where E_M = expected return on the market portfolio
R = risk free rate of return
σ_M = standard deviation of returns from the market

It indicates the additional expected return that the market requires for accepting an additional unit of risk as measured by the standard deviation of the market returns.

$$E_M = [25{,}000 \times (.2 \times (-1\%) + .4 \times 3\% + .4 \times 6\%)$$
$$+ 75{,}000 \times (.2 \times (-2\%) + .4 \times 5\% + .4 \times 8\%)] \div 100{,}000$$
$$= 4.45\%$$

$$\sigma_M^2 = [(\frac{25{,}000 \times (-1\%) + 75{,}000 \times (-2\%)}{100{,}000} - 4.45\%)^2 \times 0.2 +$$
$$(\frac{25{,}000 \times (3\%) + 75{,}000 \times (5\%)}{100{,}000} - 4.45\%)^2 \times 0.4 +$$
$$(\frac{25{,}000 \times (6\%) + 75{,}000 \times (8\%)}{100{,}000} - 4.45\%)^2 \times 0.4]$$
$$= 3.38\%^2$$

Market price of risk = $\dfrac{E_M - r}{\sigma_M} = \dfrac{4.45\% - 3\%}{3.38\%} = 42.9\%$

Total Marks: 9

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