Institute of Actuaries of India

Subject CT4 - Models

May 2008 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

1.

- Develop a well defined set of objectives that need to be met by the modeling process
- Plan the modeling process and how the model will be validated
- Collect and analyse the necessary data for the model
- Define the model by capturing the essence of the real world system
- Involve the experts on the real world system so as to get feedback on the validity of the conceptual model
- Decide on the type of model to be used for implementation of the model
- Write the computer program for the model
- Debug the program to make sure it performs the intended operations in the model
- Test the reasonableness of the output from the model
- Review and carefully consider the appropriateness of the model in the light of small changes in input parameters
- Analyse the output from the model
- Communicate and document the results and the model

[3]

2.

(a) White noise is a stochastic process that consists of a set of independent and identically distributed random variables. The random variables can be either discrete or continuous and the time set can be either discrete or continuous.

(b) A Poisson process with rate λ is a continuous-time integer-valued process N_t, $t \geq 0$ with the following properties:
(i) $N_0 = 0$
(ii) N_t has independent increments
(iii) N_t has Poisson distributed stationary increments:

$$P[N_t - N_s = n] = \frac{[\lambda(t-s)]^n e^{-\lambda(t-s)}}{n!} \quad , s<t, n=0,1,....$$

[3]

3.

(i) θ_x deaths *x* nearest b'day during period of investigation (POI)
Rate interval is life year (x - ½, x + ½)

(ii) Define $P_x^{'}(t)$ census at time t after start of POI of these aged *x* nearest at time *t*. So Central *E* to *R*, E_x^c, corresponding to θ_x , is given by

$$E_x^c = \int_{t=0}^{t=3} P_x^{'}(t).dt$$

$$\cong P_x^{'}(\frac{1}{2}) + P_x^{'}(1\frac{1}{2}) + P_x^{'}(2\frac{1}{2})$$

If $P_x^{'}(t)$ is linear in t over (0,1) (1,2) and (2,3).

Now the censuses given are *Px*(*t*) where *x* is *x* last birthday at time *t*. If birthdays are uniformly distributed over CY then

$$P_x^{'}(t) = ½\ P_{x-1}^{t} + ½\ P_x^{t}$$

So

$$E_x^c = ½\ [\ P_x(\frac{1}{2}) + P_x(1\frac{1}{2}) + P_x(2\frac{1}{2}) + P_{x-1}(\frac{1}{2}) + P_{x-1}(1\frac{1}{2}) + P_{x-1}(2\frac{1}{2})\ ]$$

(iii) and $m_x^0 = \frac{\theta_x}{E_x^c}$ estimates $m_{x-1/2}$, assuming date of births are uniformly distributed over calendar years.

[7]

4.

(a)At each age there will be a different sample size/exposed to risk, *Ex* . This will usually be largest at ages where many term polices are sold e.g.25 to 50 and smaller at other ages. The estimation procedure should pay more attention to ages where there are lots of data. These ages should have a greater influence on the choice of α and λ than other ages. So weights *bx* $\propto$ *Ex*

Suitable choice is *wx* = *[var(* $q_x^{\wedge}$ *)]* $^{-1}$

$$= \frac{E_x}{q_x(1-q_x)}$$

$$\cong \frac{E_x}{q_x} \text{ as } q_x \cong 10^{-2}$$

These weights can be estimated by $\frac{E_x}{q_x^{\wedge}} = \frac{E_x^2}{\theta_x}$

(b) The graduated rates q_x^0 are a linear function of the rates in the standard table q_x^s. The standard table rates will already be smooth. Further suitability of the formula can be investigated by applying the statistical tests.

(c)Smoothness is based on the size of the third differences of the graduated rates $\Delta^3 q_x^0$, which because the relationship is linear will be equal to b $\Delta^3 q_x^s$. $\Delta^3 q_x^s$ will already be acceptably small because the standard table rates will already be smooth.

[7]

5.

Choose a period of investigation from time 0 to time T, where T is a whole number of calendar years (say about 4) and 0 corresponds to the start of a calendar year.

x = age next birthday on policy anniversary before death

Let $\sum \vartheta_x$ be total deaths labelled x in all calendar years during period of investigation.

Let $P_x(t)$ be a census at time t after start of period of investigation of those lives having age label x at time t.

Then,

$$E_x^c = \int_{t=0}^{t=T} P_x(t).dt$$

$$= \tfrac{1}{2} P_x(0) + \sum_{t=1}^{t=T-1} P_x(t) + 1/2 P_x(T)$$

Assuming $P_x(t)$ varies linearly with t over each calendar year.

Then

$$\hat{\mu_x} = \frac{\sum \theta_x}{E_x^c} \text{ estimates } \mu_x$$

Policy year rate interval, average x - ½ at start assuming birthdays are uniformly distributed over the policy year, and that the force of mortality is constant over each year of age.

[7]

6)

(a) Hazard Rate, $h(x) = \underset{h \to 0+}{Lim} \frac{P[x < X \le x+h | X > x]}{h}$

Integrated Hazard $H(x) = \int_{u=0}^{u=x} h(u).d(u) = -\ln(S(x))$

$= -\ln P[X > x]$

where $S(x) = P[X > x]$

(b) From (a)

$$H(x)=\int_{u=0}^{u=x}\alpha\lambda u^{\alpha-1}.du$$

$$= \frac{\alpha}{\alpha}\lambda u^{\alpha}\Big|_{u=0}^{u=x}$$

$$= \lambda x^{\alpha} \qquad x>0$$

(c) $h(x) = (\alpha_0 + \alpha_1 z_1)(\lambda_1 z_1 + \lambda_2 z_2)x^{\alpha_0+\alpha_1 z_1-1}$

$$\frac{h(x|z)}{h(x|z^*)} = \frac{(\alpha_0+\alpha_1 z_1)(\lambda_1 z_1+\lambda_2 z_2)x^{\alpha_0+\alpha_1 z_1-1}}{(\alpha_0+\alpha_1 z^*_1)(\lambda_1 z^*_1+\lambda_2 z^*_2)x^{\alpha_0+\alpha_1 z_1^*-1}}$$

which is not in general independent of x, so hazards are not proportional.

If $\alpha_1 = 0$ then $\frac{h(x|z)}{h(x|z^*)} = \frac{\alpha_0+(\lambda_1 z_1+\lambda_2 z_2)x^{\alpha_0-1}}{\alpha_0+(\lambda_1 z_1^*+\lambda_2 z_2^*)x^{\alpha_0-1}}$

$$= \frac{\lambda_1 z_1+\lambda_2 z_2}{\lambda_1 z_1^*+\lambda_2 z_2^*}$$

which is independent of x, so the hazards are proportional **[8]**

7)

Group 1

j	t_j	d_j	n_j	$\lambda_j(=\frac{d_j}{n_j})$	1- λ_j	F(t)	$\frac{d_j}{n_j(n_j-d_j)}$	V[F(t)]
1	143	1	19	0.0526	0.9474	0.0526	0.0029	0.0026
2	165	1	18	0.0556	0.9444	0.1053	0.0033	0.0050
3	188	2	17	0.1176	0.8824	0.2105	0.0078	0.0087
4	190	1	15	0.0667	0.9333	0.2632	0.0048	0.0102
5	192	1	14	0.0714	0.9286	0.3158	0.0055	0.0114
6	206	1	13	0.0769	0.9231	0.3684	0.0064	0.0122
7	208	1	12	0.0833	0.9167	0.4211	0.0076	0.0128
8	212	1	11	0.0909	0.9091	0.4737	0.0091	0.0131
9	216	1	10	0.1000	0.9000	0.5263	0.0111	0.0131
10	220	1	8	0.1250	0.8750	0.5855	0.0179	0.0131
11	227	1	7	0.1429	0.8571	0.6447	0.0238	0.0126
12	230	1	6	0.1667	0.8333	0.7039	0.0333	0.0117
13	244	1	5	0.2000	0.8000	0.7632	0.0500	0.0103
14	246	1	3	0.3333	0.6667	0.8421	0.1667	0.0087
15	265	1	2	0.5000	0.5000	0.9211	0.5000	0.0053
16	303	1	1	1.0000	0.0000	1.0000		0.0000

Group 2

j	t_j	d_j	n_j	$\lambda_j (= \frac{d_j}{n_j})$	1- λ_j	F(t)	$\frac{d_j}{n_j(n_j - d_j)}$	V[F(t)]
1	142	1	22	0.04545	0.95455	0.04545	0.0022	0.00197
2	157	1	21	0.04762	0.95238	0.09091	0.0024	0.00376
3	163	1	20	0.05000	0.95000	0.13636	0.0026	0.00535
4	198	1	19	0.05263	0.94737	0.18182	0.0029	0.00676
5	205	1	17	0.05882	0.94118	0.22995	0.0037	0.00817
6	232	3	16	0.18750	0.81250	0.37433	0.0144	0.01104
7	233	4	13	0.30769	0.69231	0.56684	0.0342	0.01171
8	239	1	9	0.11111	0.88889	0.61497	0.0139	0.01131
9	240	1	8	0.12500	0.87500	0.66310	0.0179	0.01068
10	261	1	7	0.14286	0.85714	0.71123	0.0238	0.00984
11	280	2	6	0.33333	0.66667	0.80749	0.0833	0.00746
12	295	2	4	0.50000	0.50000	0.90374	0.2500	0.00418
13	323	1	2	0.50000	0.50000	0.95187	0.5000	0.00220

Comments

Though there is a slight indication from the data that the group 2 has more impact from a particular type of cancer after exposure to a particular carcinogen, however from the variances, this is not statistically significant.

[9]

8.

(i) The three assumptions underlying the simple two-state model are:

(a) The probabilities that a life at any given age will be found in either state at any subsequent age depend only on the ages involved and on the state currently occupied.

(b) ${}_{dt}q_{x+t} = \mu_{x+t}dt + o(dt)$ (t≥0)

(c) For each integer x, μ_{x+t} takes a constant value μ for 0≤t≤1

(ii)

(a) Exponential distribution in each case with σ in H and ρ in S.

(b) The time spent in state *H* before the next visit to *S* has mean σ^{-1}.

Therefore a reasonable estimate for σ is the reciprocal of the mean length of each visit: $\hat{\sigma}$= (Number of transitions from *H* to *S*)/(Total time spent in state *H* up until the last transition from *H* to *S),* although it would be equally valid to use the Maximum Likelihood Estimator, which is (Number of transitions from *H* to *S*)/(Total time spent in state *H).*

Similarly for $\hat{\rho}$

(c) Testing whether the successive holding times are independent exponential variables would be best, and any procedure which does test this is acceptable. Something like using the χ^2 goodness-of-fit test on the even-numbered holding times, then again on the odd-numbered ones, springs to mind, but there may be other, equally reasonable, answers.

(iii)

(a) For a time-inhomogeneous model the transition rates σ and ρ are functions of *t.*

It is certainly possible to improve the fit by using a time-inhomogeneous model in this instance.

(b) If the age profile is represented by a density function *f(a);* then the overall average rate at which a healthy employee falls sick is

$\sigma = \int f(a)\sigma(a)da$, roughly constant for all *t.* The same of course applies to the overall average rate of recovery.

[11]

9)

	Age	Initial exposed to risk	Actual no of deaths	Graduated mortality rates	Expected deaths	Standard deviation	Standardized deviation	χ^2
	x	E_x	θ_x	$q^{\wedge}_x$	$E_x\ q^{\wedge}_x$	$\sqrt{E_x}\,q^{\wedge}_x(1-q^{\wedge}_x)$	(3)-(5) /(6)	
1	**12**	**601250**	**161**	**0.00028**	168.35	12.9732	-0.566554	0.320984
2	**13**	**647273**	**205**	**0.00033**	213.60	14.6126	-0.588531	0.346369
3	**14**	**702000**	**260**	**0.00038**	266.76	16.3297	-0.41397	0.171371
4	**15**	**765000**	**344**	**0.00043**	328.95	18.1331	0.8299749	0.688858
5	**16**	**836458**	**418**	**0.00048**	401.50	20.0327	0.8236552	0.678408
6	**17**	**916642**	**506**	**0.00053**	485.82	22.0355	0.9157957	0.838682
7	**18**	**760763**	**463**	**0.00059**	448.85	21.1798	0.6680885	0.446342
8	**19**	**602909**	**388**	**0.00066**	397.92	19.9413	-0.497459	0.247465
9	**20**	**446635**	**318**	**0.00074**	330.51	18.1732	-0.688376	0.473862
10	**21**	**367289**	**296**	**0.00083**	304.85	17.4527	-0.507085	0.257135
11	**22**	**290086**	**251**	**0.00093**	269.78	16.4173	-1.143912	1.308535
			3610		3,616.8		-1.1684	5.7780

χ^2 test :

The resulting χ^2 value is 5.78, far below 16.92, the upper 5% point of χ^2_9
2 - well below even the mean, 9. We should use at most (11-2=) 9 degrees of freedom because of the two parameters estimated.

However, reducing the degrees of freedom even as low as 2 (upper 5% point = 5.991) would still yield a non-significant result.

This test certainly does not reveal any departure from the null hypothesis.

Individual standardized deviations:

<-2	**0**	**expected**	**2%**
-2 to -1	**1**	**expected**	**14%**
-1 to 0	**6**	**expected**	**34%**
0 to 1	**4**	**expected**	**34%**
> 1	**0**	**expected**	**16%**

i.e. concentrated somewhat closer to zero than expected on the basis of a Standard normal distribution. But not unsatisfactory.

Cumulative deviations (over the whole age range):

$\sum(\theta_x - E_x q_x)$ = -6.89 $\sum E_x p_x q_x \approx \sum E_x q_x$ = 3616.8

-6.8 is (in modulus) much less than $2(3616.8)^{1/2} \approx 120$.
Clearly this is not significant, (but test is not rigorous because the process of graduation constrains cumulative deviations to be close to zero).

Signs test:
There are 4 plus signs out of 11. If *Binomial* (11, ½),
$P(N \le 4) \ge P(N = 4) = 0.1611$ This is clearly not significant.

Grouping of signs

There are $n1 = 4$ positive deviations and $n2 = 7$negative deviations. There is only one group of positive deviations.
Under $H0$ the probability of one group of positive deviations is $8 \Big/ \binom{11}{4}$

Ie 8/330 = 0.024. This is in fact the *p*-value, the probability of one or fewer groups of positive deviation signs. This is significant at 5% level.

Serial correlation test (as an alternative to grouping of signs):

Since $r_1\sqrt{11}$ is approximately standard normal, and $r_1\sqrt{11} = 2.10 > 1.65$, there is significant positive serial correlation (at the 5% level).

Comment: Fidelity to data appears to be very satisfactory apart from the excessive clumping of the deviations of the same sign. The graduated Rates are lower than the crude rates in the middle of the age range, and Higher at either end. This suggests that, over the age range 12-22, the Standard table used differs slightly in shape from the experience

[12]

10.

(i) The Chapman – Kolmogrov equations are

$P_{ij}(s,t) = \sum_{k \in s} P_{ik}(s,u) P_{kj}(u,t)$

To obtain the forward equations we differentiate with respect to t and evaluate at u=t;

$$\frac{\partial}{\partial t} P_{ij}(s,t) = \sum_{k \in s} \left[P_{ik}(s,u) \left(\frac{\partial}{\partial s} P_{kj}(u,t) \right) \right]_{u=t} = \sum P_{ik}(s,t) \mu_{kj}(t)$$

Similarly the backward equations are obtained by differentiating with respect to s and setting u=s ;

$$\frac{\partial}{\partial s} P_{ij}(s,t) = \sum_{k \in s} \left[\left(\frac{\partial}{\partial s} P_{ik}(s,u) \right) P_{kj}(u,t) \right]_{u=s} = -\sum_{k \in s} \mu_{ik}(s) P_{kj}(s,t)$$

We now need to explain where the minus sign in the RHS comes from.
The definition of the transition rates is such that ;

$$P_{ik}(s, s+h) = \delta_{ik} + h\mu_{ik}(s) + o(h)$$

Or equivalently;

$$P_{ik}(s-h, s) = \delta_{ik} + h\mu_{ik}(s-h) + o(h)$$

Rearranging this gives:

$$\mu_{ik}(s-h) = \frac{P_{ik}(s-h,s) - \delta_{ik} - o(h)}{h}$$

Now taking the limit of both sides as h->0 and nothing that $P_{ik}(s,s)=\delta_{ik}$, we get

$$\mu_{ik}(s) = -\lim_{h \to 0} \frac{P_{ik}(s-h,s) - P_{ik}(s,s) - o(h)}{-h} = -\left[\frac{\partial}{\partial s} P_{ik}(s,t) \right]_{t=s}$$

(ii) $P'_{0,0}(t) = \mu P_{0,1}(t) - \lambda P_{0,0}(t)$ or a more general form such as

$$P'_{0,0}(t) = \sum P_{0,k}(t) \sigma_{k,0}$$

(iii) Since $P_{0,1}(t) = 1 - P_{0,0}(t)$,

We have $P'_{0,0}(t) = \mu(1 - P_{0,1}(t)) - \lambda P_{0,0}(t)$. Any solution method will do,

e.g. $\frac{d}{dt}\left[e^{(\lambda+\mu)t} P_{0,0}(t) \right] = \mu e^{(\lambda+\mu)t}$ solved by $P_{0,0}(t) = \frac{\mu}{\lambda+\mu} + Ce^{-(\lambda+\mu)t}$

with C being determined by the fact that $P_{0,0}(0)=1$.

(iv) $E_0 O_t = E_0 \int_0^t I_s ds = \int_0^t E_0 I_s ds = \int_0^t P_{0,0}(s) ds$

$$= \frac{\mu}{\lambda+\mu} t + \frac{\lambda}{(\lambda+\mu)^2}(1 - e^{-(\lambda+\mu)t})$$

(v) Since the process must be in state 0 or state 1 at all times, the solution is just

$$t - E_0 O_t = \frac{\lambda}{\lambda+\mu} t - \frac{\lambda}{(\lambda+\mu)^2}(1 - e^{-(\lambda+\mu)t})$$

(vi) (a) Assuming a member who is initially healthy, expected outgoings (including expenses) by time t and expected income by time t, are respectively

$$\gamma t + \beta\left(\frac{\lambda}{\lambda+\mu} t - \frac{\lambda}{(\lambda+\mu)^2}(1 - e^{-(\lambda+\mu)t})\right)$$

And $\alpha\left(\frac{\mu}{\lambda+\mu} t + \frac{\lambda}{(\lambda+\mu)^2}(1 - e^{-(\lambda+\mu)t})\right)$

In the long run then as t->∞, we require αμ=βλ+γ(λ+μ) to break even

(b) The assumptions required are that the rate of becoming ill and rate of recovery from illness are constant

(c)This will certainly not be true of any individual member but, if membership is large and the age and health profiles of the members are constant by virtue of a constant influx of new members, it may be a reasonable approximation.

[16]

11).

(i) There is an explicit dependence on the past behavior of Y_j, $j \le n$ in the probability distribution of Y_{n+1}; hence the Markov property does not hold.

On the other hand

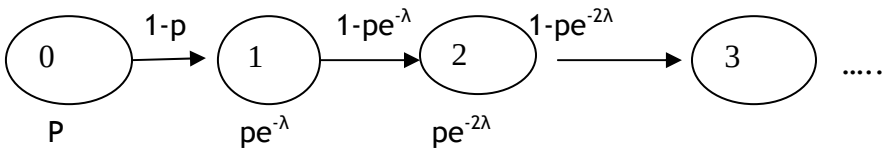
$P[X_{n+1}=j / X_1=i_1, X_2=i_2, \ldots\ldots X_{n-1}=i_{n-1}, X_n=i_n]$

$= P[Y_{n+1}=j-i / Y_1=i_1, Y_2=i_2-i_1, \ldots\ldots Y_{n-1}=i_{n-1}-i_{n-2}, Y_n=i_n-i_{n-1}]$

$$= \begin{cases} pe^{-\lambda i} if\ (j-i)=0 \\ 1-pe^{-\lambda i} if\ (j-i)=1 \end{cases}$$

This is independent of $i_1, i_2 \ldots\ldots i_{n-1}$.

(ii) Transition graph



Transition matrix:

$$\begin{pmatrix} p & 1-p & & \\ & pe^{-\lambda} & 1-pe^{-\lambda} & 0 \\ & & pe^{-2\lambda} & 1-pe^{-2\lambda} \\ 0 & & .. & .. \end{pmatrix}$$

(iii)

(a) Chain is time-homogenous since transition probabilities calculated in (i) do not depend on time n.

(b) It is not irreducible since the number of accidents can never go down

(c) There are no recurrent states, hence there can be no stationary distribution. Alternatively, a stationary distribution, π, if it exists, must obey

$\Pi_0 p = \Pi_0$
$\Pi_{0(1-}p) + \Pi_1 pe^{-\lambda} = \Pi_1$
$\Pi_{1(1-}\ pe^{-\lambda}) + \Pi_2 pe^{-2\lambda} = \Pi_2$
;

Since p<1 we have $\Pi_0=0$ and $\Pi_1=0$ etc. Hence no stationary probability distribution exists.

(iv) No new accident;

$$(pe^{-j\lambda})^n = p^n e^{-nj\lambda}$$

(v) (a) Maximum likelihood would be very easy in this case; choose λ and p to maximize
$\Pi\{(pe^{-\lambda x_k})^{1-y_k}(1-\ pe^{-\lambda x_k})^{y_k}\}$.

(b) Change the model to:

$P[Y_{n+1}=0/\ Y_1=y_1,\ Y_2=y_2,\ \ldots\ldots\ Y_n=y_n] = pe^{-\lambda(xn,n)}$

Then test the hypothesis that λ(x,n) =λ(x) for all n.

[17]

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