Institute of Actuaries of India

Subject CT1 – Financial Mathematics

May 2008 Examination

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1

Q.1 Indicative solution

(i) $A(0,10) = \text{Exp} \int_0^{10} (0.007t + 0.0003t^2)\, dt$

$= \text{Exp}\, [0.007\, t^2/2 + 0.0003\, t^3/3]_0^{10}$

$= \text{Exp}\, (0.35 + 0.1) = \text{Exp}\, (0.45)$

$= 1.56831$

Thus, accumulation of Rs. 500 is Rs. 784.155

(ii) i is such that $(1+i)^{10} = 1.56831$

$\Rightarrow i = 0.04603$

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Solution 2

Q.2 Indicative solution

(i) Present value of annuity (a)

$= 5000\, a^{(12)}_{\overline{15}|} + 5000/12 \times v^{15} \times v^{1/12}$

$= 5000\, i/i^{(12)} a_{\overline{15}|} + 5000/12 \times v^{15} \times v^{1/12}$

@ 8% pa

$= 5000 \times 1.036157 \times 8.5595 + 5000/12 \times 0.315242 \times 0.993607$

$= 44344.9292 + 130.5110$

$= 44475.4402$

Present value of annuity (b)

$= 6500\, a^{(4)}_{\overline{10}|} = 6500\, i/i^{(4)} a_{\overline{10}|}$ @ 8% pa

$= 6500 \times 1.029519 \times 6.7101$

$= 44903.1404$

Please note: if a candidate answers assuming that payment on 1Jan 2005 is to be included then give credit for the same. In such a case,

PV of annuity (a) will be 5000/12+ 44475.4402= 44892.1068

PV of annuity (b) will be 6500/4+ 44903.1404 = 46528.1404

(ii) Let A be the amount of revised annuity payable half-yearly

Present value of revised annuity

= PV of annuity (a) plus PV of annuity (b)

$A\, v^{2/12}\, \ddot{a}^{(2)}_{\overline{20}|}$ = 44475.4402 + 44903.1404

$A\, v^{1/6}\, i/d^{(2)}\, a_{\overline{20}|}$ = 89378.5806

A x 0.987255 x 1.059615 x 9.8181 = 89378.5806

=> A = Rs. 8702.19

Solution 3

Q.3 Indicative solution

The present value of an investment of 1due in 3 months time at rate of i per annum effective $=1/(1+i)^{3/12}$

The present value of the same investment at simple discount rate of 7% per annum = (1-3 x 0.07/12)

$1/(1+i)^{3/12}$ = (1-3 x 0.07/12)

=> i = (1-3 x 0.07/12)(-12/3) –1

= 7.3173 %

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Solution 4

Q.4 Indicative solution

(i) The money-weighted rate of return is found by solving equation of value:

$1{,}000{,}000\,(1+i) + 200{,}000\,(1+i)^{3/4}$ = 1,260,000

$100\,(1+i) + 20\,(1+i)^{3/4}$ = 126 ------equation(1)

Using an approximation:

100 (1+i) + 20 (1+ 3/4i) = 126

=> 115 i = 6

=> i = 5.217%

Using trial and improvement:

At 5%, the left-hand side of equation (1)= 125.745

At 5.5%, the left-hand side of equation (2)= 126.319

Interpolating between these value we get:

(i –5%)/(5.5%-5%)=(126 – 125.745)/(126.319-125.745)

=> i = 5.22%

(ii) The time-weighted rate of return is found from the equation:

1+i = (116-20)/100 x 126/116 = 1.0428

=> i = 0.428

So the TWRR is 4.28%

(iii) The linked quarterly rate of return can be found by combining the "growth factors" for each quarter:

1+i = (116-20)/100 x 120/116 x 128/120 x 126/128 = 1.0428

=> i = 0.428

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Solution 5

Q.5 Indicative solution

(i) Eurobonds:

- A form of corporate or government borrowing
- Usually unsecured
- Pay regular coupon and are redeemable at par
- Issued in many different currencies, eg Yen, DM, sterling
- Traded internationally through banks, and not in traditional market
- Yield depends on risk and marketability, but typically be slightly lower than conventional unsecured loan stocks of the same issuer

(ii) Index-linked government bonds:

- Payment guaranteed by the government

- Normally coupon and capital payments both indexed to increases in a given price index with a lag
- Low volatility of return and low expected real return.
- More or less guaranteed real return if held to maturity (can vary due to indexation lag)
- Nominal return is not guaranteed.
- Fairly liquid but lower than government bonds

(iii) One party agrees to pay to the other a regular series of fixed amounts for a certain term.
In exchange the second party agrees to pay a series of variable amounts based on the level of a short-term interest rate.

(iv) The counterparty faces *market risk*, which is the risk that market conditions will change so that the present value of the net outgo under the agreement increases.

The counterparty also faces *credit risk*, which is the risk that the other counterparty will default on its payments

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Solution 6

Q.6 Indicative solution

(i) For determining the price, ignore tax on coupon as yield given is gross redemption yield
Price paid by first investor is P_1

$$P_1 = 4\, a^{(2)}_{\overline{15}|} + 100\, v^{15} \quad @\ 5\%\ pa$$

$$= 4\, i/i^{(2)}\, a_{\overline{15}|} + 100\, v^{15}$$

$$i/i^{(2)} = 1.012348$$

$v^{15} = 0.48102$

$a_{\overline{15}|} = 10.3797$

$P_1 \quad 4 \times 1.012348 \times 10.3797 + 100 \times 0.48102$

$= 90.1335$

(ii) $(1+ i^{(2)}/2)^2 = 1.06 \Rightarrow i^{(2} = 0.059126$

$g (1-t_1) = 0.04 \times 0.75 = 0.03$

$i^{(2} > g (1-t_1)$

=> there is capital gain on the contract

Price paid by second investor is P_2

$P_2 = 4 \times 0.75\, a^{(2)}_{\overline{7}|} + 100 v^7 – 0.4 \times (100-P_2) v^7$ @ 6% pa

$P_2 (1-0.4 v^7) = 3 \times i/i^{(2)} a_{\overline{7}|} + 0.6 \times 100 v^7$

At 6% pa, $i/i^{(2)} = 1.014782$, $v^7 = 0.66506$, $a_{\overline{7}|} = 5.5824$

$P_2 (1-0.4 \times 0.66506)= 3 \times 1.014782 \times 5.5824 + 60 \times 0.66506$

$P_2 = 77.5207$

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Solution 7

Q.7 Indicative solution

(i) The three conditions are

- The value of assets at the starting rate of interest is equal to the value of the liabilities
- The volatilities of the asset and liability cashflow series are equal
- The convexity of the asset cashflow series is greater that the convexity of the liability cashflow series

(ii) Let A and B be the nominal (or maturity) amount of 15-year and 23 –years zero coupon bonds respectively.

From the first condition of immunization

Value of assets = vale of liabilities

$A \times v^{15} + B \times v^{23} = 5 v^{17} + 8 v^{20}$ @ 8%pa

$A + B \times v^{8} = 5 v^{2} + 8 v^{5}$

$= 5 \times 0.85734 + 8 \times 0.68058$

$A + 0.54027 B = 9.73134$ -----------equation (1)

The second condition could be replaced by

DMT of assets = DMT of liabilities i.e.

$V'_A(i_0) = V'_L(i_0)$

$V'_A(i_0) = -A \times 15 v^{16} - B \times 23 v^{24}$

$V'_L(i_0) = -5 \times 17 v^{18} - 8 \times 20 v^{21}$

Thus,

$-A \times 15 v^{16} - B \times 23 v^{24} = -5 \times 17 v^{18} - 8 \times 20 v^{21}$

$15A + 23 B v^{8} = 85v^{2} + 160 v^{5}$

$15 A + 23 B \times 0.54027 = 85 \times 0.85734 + 160 \times 0.68058$

$15A + 12.42621 B = 181.7667$ --------equation (2)

Multiply (1) by 15 and subtract from equation (2), we get

$B \times (12.42621 – 15X\ 0.54027) = (181.7667 – 15 \times 9.73134)$

$B = 8.2821$

Substituting value of B in equation (1), we get

$A = 9.73134 – 0.54027 \times 8.2821$

$= 5.25677$

The manager will invest Rs. $5.25677v^{15} = 1.65715$ million in 15-year zero-coupon bond

And Rs. $8.2821v^{23} = 1.41057$ in 23-year zero-coupon bond.

(iii) Limitations of Immunization theory: (any four from the list)

- It may be necessary to rebalance the portfolio once interest rates have changed

- There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known
- Assets may not exist to provide the necessary overall asset volatility to match the liability volatility
- The theory relies upon small changes in interest rates. The fund may not be protected against large changes
- The theory assumes a flat yield curve and requires the same change in interest rates at all terms. In practice, this is rarely the case
- Immunization removes the likelihood of making large profits.

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Solution 8

Q.8 Indicative solution

Let i be the effective real rate of return per annum.

Take present value of all cash flows at purchase date

The 1^{st} dividend paid at the end of 1^{st} year in nominal term = 6

The real present value of 1st dividend at purchased date = $6/1.05\ v$

Where v is calculated at real rate of interest i pa

The 2^{nd} dividend paid at the end of 2^{nd} year in nominal term = 6×1.08

The real present value of 2^{nd} dividend = $6 \times (1.08)/1.05^2 v^2$

The 3^{rd} dividend paid at the end of 3^{rd} year in nominal term = 6×1.08^2

The real present value of 3^{rd} dividend = $6 \times 1.08^2/1.05^3 v^3$

Continuing this way,

The real present value of the last coupon = $6 \times 1.08^4/1.05^5 v^5$

And real present value of the selling proceed = $150 / 1.05^5 \ v^5$

Thus, equation of value is:

$6/1.05\ v + 6 \times (1.08) / 1.05^2\ v^2 + 6 \times 1.08^2/1.05^3\ v^3$ --------+ $150 / 1.05^5\ v^5 = 100$

$6/1.08(1.08/1.05\ v + 1.08^2/1.05^2\ v^2 \ldots\ldots + 1.08^5/1.05^{5)}\ v^5)$

$+ 150/1.08^5 \times 1.08^5/1.05^5\ v^5 = 100$

Assume $1.08/1.05\ v = v`$ at rate of return j pa

Where $1.08/(1.05 \times (1+i)) = 1/(1+j)$

Then, the above equation of value can be written as

$6/1.08(v` + v`^2 + \ldots v`^5) + 150/1.08^5 v`^5 = 100$ @ j pa

$6/1.08\ a_{\overline{5|}} + 150/1.08^5\ v`^5 = 100$

Solving this using trial and error:

Approximate j as $0.06/1.08 + (1.50^{1/5}/1.08-1) = 0.0555 + 0.00414 = 0.05969$

@ 6%, LHS = 99.6881

@ 5%, LHS = 104.0414

Therefore using the interpolation

$(j - 5\%)/(6\%-5\%) = (100 - 104.0414)/(99.6881-104.0414)$

$\Rightarrow j = 5\% + 1\% \times 4.0414/4.3533 = 5.93\%$

From $1.08/(1.05 \times (1+i)) = 1/1+j$

$1+i = (1+J)(1.08)/1.05$

$\Rightarrow i = 1.0593 \times 1.08/1.05 - 1 = 0.08956$

$= 8.956\%$

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Solution 9

Q.9 Indicative solution

(i) $J = 0.05 \times 0.4 + 0.07 \times 0.2 + 0.09 \times 0.4$

$= 0.07$

Mean accumulation $= 10{,}000 \times (1 + j)^{10}$

$= 10{,}000 \times (1.07)^{10}$

$=$ Rs. 19671.5

(ii) $s^2 = 0.05^2 \times 0.4 + 0.07^2 \times 0.2 + 0.09^2 \times 0.4 - 0.07^2$

$= 0.00522 - 0.0049$

$= 0.00032$

Var (accumulation) $= 10{,}000^2 \{(1 + 2j + j2 + s2)^{10} - (1 + j)^{20}\}$

$= 10{,}000^2 \{1 + 2 \times 0.07 + 0.07^2 + 0.00032)^{10} - (1.07)^{20}\}$

$= 10{,}000^2 \{1.14522)^{10} - (1.07)^{20}\} = 1{,}082{,}939.70$

SD(accumulation) $= (1082939.70)^{1/2} =$ Rs.1040.64

(iii) By symmetry $j = 0.07$ (as in (i))

Hence, mean (accumulation) will be same as in (i) (i.e. Rs. 196715.51).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than Rs.1040.64

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Solution 10

Q.10 Indicative solution

(i) Let S be the level monthly installment in first 10 years.

Working in unit of a quarter. The effective rate of interest per quarter will be 3%.

$3S\, a^{(3)}_{\overline{40}} + 1.20 \times 3S\, v^{40} a^{(3)}_{\overline{40}} = 1{,}500{,}000$

$S\, i/i^{(3)}\, a_{\overline{40}} + 1.20\, S\, i/i^{(3)}\, v^{40}\, a_{\overline{40}} = 500{,}000$

Where $i^{(3)} = 3[1.03^{1/3} - 1] = 0.029704$, $i/i^{(3)} = 1.00996$

$a_{\overline{40}} = 23.1148$, $v^{40} = 0.30656$

$S \times 1.00996 \times 23.1148 (1 + 1.20 \times 0.30656) = 500{,}000$

$S = 500{,}000/31.933$

$= 15{,}657.78$

So level monthly installment in first 10 years is 15,657.78 and after 10th year 18,789.34

(ii) The loan outstanding at the end of 10th years

$= 3 \times 18{,}789.34 a^{(3)}_{\overline{40}} = 56{,}368.02\ i/i^{(3)}\ a_{\overline{40}}$

$= 56{,}368.02 \times 1.00996 \times 23.1148$

$= 1{,}315{,}912.75$

The loan outstanding at the end of 15th years

$= 3 \times 18{,}789.34 \times a^{(3)}_{\overline{20}} = 56{,}368.02\ i/i^{(3)}\ a_{\overline{20}}$

$= 56{,}368.02 \times 1.00996 \times 14.8775 = 846{,}967.825$

Thus, the total capital repayment from 11th years to 15th years

$= 1{,}315{,}912.75 - 846{,}967.825 = 468{,}944.925$

The total interest payment from 11th to 15th years

= Total installments paid over the period – total capital repayment during the period

$= 60 \times 18{,}789.34 - 468{,}944.925 = 658{,}415.475$

[10]

Solution 11

Q.11 Indicative solution

(i) A forward contract is an agreement made at some time $t = 0$, say, between two parties under which one agrees to buy from the other a specified amount of an asset (denoted S) at a specified price on a specified future date.

The investor agreeing to sell the asset is said to hold a short forward position in the asset, and the buyer is said to hold a long forward position.

(ii) We first need to find the stock price, S_0 for Rs. 100 nominal. We have:

$S_0 \quad = 5 \ddot{a}^{(2)}_{\overline{\infty}} v^{4/12} + 50 v^4 \quad = 5 v^{1/3}/d^{(2)} + 50 v^4$ @ 5.5% per annum effective.

$d^{(2)} = 2(1-1.055^{-0.5}) \quad = 0.05283$

So the price is:

$S_0 = 5 \times 0.982311 / 0.05283 + 50 \times 0.807216 = 133.33$

The forward price is given by

$K_0 \quad = \quad (S_0 - I)(1+i)^5,$

Where I is the present value of income from the stock during the term and $i =$ 0.06.

$I \quad = \quad 5 \ddot{a}^{(2)}_{\overline{5}} v^{4/12} + 50 v^4$

$\quad = \quad 5 v^{1/3}\ i/d^{(2)}\ a_{\overline{5}} + 50 v^4$

$I \quad = \quad 5 \times 0.980764 \times 1.044782 \times 4.2124$

$\quad + 50 \times 0.792094$

$\quad = \quad 61.1866$

Thus, the forward price is:

$K_0 \quad = \quad (133.33 - 61.1866) \times (1.06)^5$

$\quad = \quad 96.5441$

Since this is for £100 nominal, the forward price for the stock in the question is:

4 x 96.5441= 386.1764

[8]

Solution 12

Q.12 Indicative solution

(i) (a) An equation of value expresses the equality of the present value of positive and negative (or incoming and outgoing) cash flows that are connected with an investment project, investment transaction etc.

(i) (b) The discounted payback period is the point at which net revenues from the project can be used to repay all loans necessary to finance the project outgoings accumulated with interest

(ii) **Present value of costs**

Working in units of month, the effective rate of interest per month is 1.5%

The Present value of costs at 1 July 2008 at the rate of 1.5% per month effective

Initial Setup costs and initial costs of enrolling the players

= 390

Cost of building infrastructure

$= 20\ \bar{a}_{\overline{6}|} = 20\ i/\delta\ a_{\overline{6}|}$

$= 20 \times 1.007481 \times 5.6972$

= 114.7964

Running cost of the Cost of event

$= v^6\ [v + 1.005\ v^2 + 1.005^2\ v^3 + \text{------} + 1.005^{59}\ v^{60}]$

$= v^6/1.005\ [1.005v + 1.005^2\ v^2 + 1.005^3\ v^3\ \text{---}$

$\text{---} + 1.005^{60}\ v^{60}]$

$= v^6/1.005\ (a_{\overline{60}|}$ at j %)

Where 1.005/1.015 = 1/(1+j)

=> j = 1%

and v^6 at 1.5%= 0.91454 ,

$a_{\overline{60}|}$ at 1 % = 44.9550

= 0.91454/1.005 x 44.9550

= 40.9086

Present value of revenue (excluding sales proceeds)

Present value of revenue at 1 July 2008 (working in units of month and at interest rate of 1.5% per month)

Sale of television rights:

$25\ \bar{a}_{\overline{6}|} = 25\ i/\delta\ a_{\overline{6}|}$

$= 25 \times 1.007481 \times 5.6972$

$= 143.4955$

Other revenue from sale of merchandise, marketing rights, tickets, etc.:

$= 4\ v^6 [v + 1.01\ v^2 + 1.01^2\ v^3 + \text{------} + 1.01^{59}\ v^{60}]$

$= 4\ v^6/1.01\ [1.01v + 1.01^2\ v^2 + 1.01^3\ v^3 + \text{------} + \quad 1.01^{60}\ v^{60}]$

$= 4\ v^6/1.01\ (a_{\overline{60}|}\ \text{ at } j`\ \%)$

Where $1.01/1.015 = 1/(1+j`)$

$\Rightarrow j` = 0.5\%$

v^6 at $1.5\% = 0.91454$,

and $a_{\overline{60}|}$ at $0.5\ \% = 51.7256$

$= 4 \times 0.91454/1.01 \times 51.7256$

$= 187.3405$

Let S be the required sale price

Required equation of value

PV of revenue = PV of costs

$S\ v^{66} + 143.4955 + 187.3405 = 390 + 114.7964$

$+ 40.9086$

$0.37432\ S = 545.705 - 330.836 = 214.869$

$S = 574.025$ crores

[14]

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