INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

14th May 2008

Subject CT4– Models

Time allowed: Three Hours (10.00 am – 13.00 Hrs)

Total Marks: 100

INSTRUCTIONS TO THE CANDIDATES

1. ***Do not write your name anywhere on the answer sheet/s. You have only to write your Candidate's Number on each answer sheet/s.***
2. ***Mark allocations are shown in brackets.***
3. ***Attempt all questions, beginning your answer to each question on a separate sheet.***
4. ***Fasten your answer sheets together in numerical order of questions. This, you may complete immediately after expiry of the examination time.***
5. ***In addition to this paper you should have available graph paper, Actuarial Tables and an electronic calculator.***

Professional Conduct:

"It is brought to your notice that in accordance with provisions contained in the Professional Conduct Standards, If any candidate is found copying or involved in any other form of malpractice, during or in connection with the examination, Disciplinary action will be taken against the candidate which may include expulsion or suspension from the membership of IAI."

Candidates are advised that a reasonable standard of handwriting legibility is expected by the examiners and that candidates may be penalized if undue effort is required by the examiners to interpret scripts.

AT THE END OF THE EXAMINATION

Please return your answer sheet/s and this question paper to the supervisor separately

Q1) Discuss briefly the key steps involved in a modeling process. **[3]**

Q2) Define each of the following examples of a stochastic process:

(a) White noise (1)

(b) Poisson process (2)

[3]

Q3) A mortality investigation is carried over out over the three calendar years viz. 2005, 2006 and 2007. Based on this investigation, following data have been tabulated:

(a) Censuses of numbers of lives aged *x* last birthday on 30 June 2005, 30 June 2006 and 30 June 2007, denoted by $p_x(½)$, $p_x(1½)$, $p_x(2½)$ respectively; and

(b) Count of number of deaths during the period of the investigation aged *x* nearest birthday at the date of death, denoted by θ_x.

i) State the rate interval that you would use for this investigation with appropriate reason. (1)

ii) Derive a formula for the estimation of central rates of mortality. (5)

iii) Determine the age to which the estimated mortality rate relates. State clearly any assumptions that you make. (1)

[7]

Q4) A large life office has investigated its recent mortality experience of its term assurance policyholders. The life office does this by estimating the initial rate of mortality q_x.

The graduated rates of mortality are given by q_x^0. The life office arrives at this by graduating the crude mortality estimates $\hat{q}_x$ with reference to a standard mortality table with rates q_x^s, so that the graduated rates are given by

$$q_x^0 = \alpha + \lambda q_x^s$$

where α and λ are constants. The estimates of α and λ will be determined by minimizing

$$\sum_x b_x (\hat{q}_x - \alpha - \lambda q_x^s)^2$$

where b_x is a suitably chosen weighting function.

(a) Explain why is it important to use the weighting function, in formula and determine an expression for a suitable choice of function for b_x (4)

(b) Describe how the suitability of the formula $q_x^0 = \alpha + \lambda q_x^s$ could be investigated. (1)

(c) Explain how the smoothness of the graduated rates, q_x^0, is ensured. (2)

[7]

Q5) A life insurance company issues only annual premium life assurance policies. The company keeps records of its life assurance policies in two files; inforce file and a claims file. For each premium paying policy, the in-force file includes the following information:

- age last birthday at the date of policy issuance
- smoking status (smoker or non-smoker)
- sex
- type of policy (temporary assurance, whole life assurance or endowment assurance)

For each age *x* (where *x* = age last birthday at date of policy issue + number of annual premiums paid since policy issuance) a count of the number of policies sub-divided by smoking status, sex and type of policy is tabulated on 1st January of each year.

For each policy for which a death claim has been paid the claim file contains the same information as the inforce file.

For each age *x* (where *x* = age last birthday at date of policy issuance + number of annual premiums paid up to the date of death) a count of death claims divided by smoking status, sex and type of policy is tabulated on each 1st January.

The company wishes to investigate the recent mortality experience of its life assurance policies

Defining suitable symbols, derive a formula for the estimation of the force of mortality using the data for an age group *x* according to the definitions given. State any assumptions that are required for your formula. State, with reasons, the age to which your estimate of the force of mortality applies. **[7]**

Q6) *X* is a random variable which measures the duration from the date of a heart Transplant until death.

(a) Express the hazard rate and the integrated hazard function at duration x, in terms of probabilities. (2)

(b) If the hazard rate at duration *x* is

$$h(x) = \alpha\lambda\, x^{\alpha-1}$$

Derive an expression for the integrated hazard, $H(x)$. (2)

(c) The hazard rate $h(x)$, as defined in part (b), varies between transplant patients in such a way that

$$\alpha = \alpha_0 + \alpha_1 z_1$$
$$\lambda = \lambda_1 z_1 + \lambda_2 z_2$$

Where α_0, α_1, λ_1, λ_2 are constants; z_1 is the age of the patient at the date of the transplant and z_2 is the patient's sex where z_2 = 0 = female, z_2 = 1 = male.

Show that these hazards are not in general proportional, but that if α_1 = 0 the hazards are proportional. (4)

[8]

Q7) A, hospital conducted some experiments on death from a particular type of cancer after exposure to a particular carcinogen and it was measured in two groups of rats. **Group 1** had a different pre-treatment régime than **Group 2**.

The time from pre-treatment to death is recorded. If a rat is still alive at the end of the experiment or it had died due to a different cause then that time is considered as "censored". A censored observation is given the value 0 in the death/censorship variable to indicate a "non-event"(= censored data).

Group's =>	2	1	2	2	1	1	1	1	1	2	2	2	1	1	1	1	1	1	1	1	2
Time	142	143	157	163	165	188	188	190	192	198	204	205	206	208	212	216	216	220	227	230	232
Censored	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	0	1	1	1	1	1
Group's =>	2	2	2	2	2	2	1	2	2	1	1	2	1	2	2	2	2	1	2	2	
Time	232	232	233	233	233	233	235	239	240	244	246	261	265	280	280	295	295	303	323	344	
Censored	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	0	

Using Kaplan- Meier estimates, determine the mean and variance of the two groups and compare the effects of the exposure to the carcinogen between the two groups. **[9]**

Q8) A company is studying the health records of its longest serving employees in order to improve its provision for health insurance. Let $X(t) = H$ if the employee is healthy at time t, otherwise $X(t) = S$. The available information includes the value of $X(t)$ for all $0 \leq t \leq T$.

(i) What are the three underlying assumptions for the simple two-state model? (3)

(ii) The company chooses to model X as a two-state time-homogeneous Markov jump process with transition rates $\sigma_{HS} = \sigma, \sigma_{SH} = \rho$

a) State the distribution of a typical holding time in state H and of a typical holding time in state S assuming the model is valid. (1)

b) Write down estimates for the parameters σ and ρ of the model in terms of quantities, which may be derived from the available data. (2)

c) Indicate one test which could be used to determine whether the data support the assumption that the Markov jump process model is suitable (1)

(iii) Having fitted the model the company discovers that the observed distribution of holding times in state H does not fit the predictions of the time-homogeneous Markov model; in particular, the mean holding time in state H between visits to state S appears to be decreasing with t.

a) Describe the principal difference between a time-inhomogeneous model and a time-homogeneous one and indicate whether a time-inhomogeneous model might provide a better fit to the observations. (2)

b) Explain why the original model could still be used if the company is large and has a roughly constant age profile. (2)

[11]

Q9) A graduation of the mortality experience in the age group 12 to 22 had been carried out and the following is the extract from the results.

Age	Initial Exposed to Risk	Actual no of deaths	Graduated mortality rate
$\boxed{x}$	$\boxed{E_x}$	$\boxed{\theta_x}$	q_x^0
12	601,250	161	0.00028
13	647,273	205	0.00033
14	702,000	260	0.00038
15	765,000	344	0.00043
16	836,458	418	0.00048
17	916,642	506	0.00053
18	760,763	463	0.00059
19	602,909	388	0.00066
20	446,635	318	0.00074
21	367,289	296	0.00083
22	290,086	251	0.00093

Perform six appropriate tests to determine whether or not the Graduation is satisfactory.

The observed value of the serial correlation coefficient (with lag one) between the ten pairs of values of the standardized deviations is 0.6344.

Comment on your results. **[12]**

Q10) Consider a time-homogeneous Markov jump process $\{X(t) : t \geq 0\}$ with two states denoted by 0, 1, and transition rates $\sigma_{0,1} = \lambda$, $\sigma_{1,0} = \mu$.

(i) Write down the Chapman-Kolmogrov equations and differentiate it to obtain the forward and backward equations. (4)

(ii) State Kolmogorov's forward equation for the probability $P_{o,o}(t)$ that X is in state 0 at time t, given that it starts in state 0. (1)

(iii) Show that $P_{o,o}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$ (3)

(iv) Let O_t denote the total amount of time spent in state 0 until time t, where

$$O_t = \int_0^t I_s \, ds \quad , \text{ where } \quad I_s = \begin{cases} 1 & if X_s = 0 \\ 0 & if X_s \neq 0 \end{cases}$$

Derive, using the result in part (ii), an expression for $E[O_t / X(0) = 0]$, the expected occupation time in state 0 by time t for the two-state continuous-time Markov chain starting in state 0. (3)

(v) Write down the expected occupation time in state 1 by time t for the two-state continuous-time Markov chain starting in state 0. (1)

(vi) A health insurance scheme labels members as "healthy" (state 0) or "unhealthy" (state 1) at any time. When in state 0, members pay contributions at rate α ; when in state 1 they receive benefit at rate β. Expenses amount to a constant γ per member per unit time.

a) Explain how the above model can be used to calculate α in terms of β and γ. (2)

b) State the assumptions which you make in applying the model. (1)

c) Discuss whether they are likely to be satisfied in practice. (1)

[16]

Q11) For a given driver, any period j is either accident free (Y_j = 0) or gives rise to one accident (Y_j = 1). The probability of having **no** accident during the next period is estimated as follows using the driver's past record (all values y_j are either 0 or 1):

$$P[Y_{n+1} = 0 / Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] = pe^{-\lambda(y_1 + y_2 + \ldots + y_n)},$$

where $0 < p < 1$, $\lambda \geq 0$.

The cumulative number of accidents suffered by the driver over the time period from 1 to n is

$$X_n = \sum_{j=1}^{n} Y_j$$

(i) Verify that the Markov property holds for the sequence $X_1, X_2, \ldots, X_n, \ldots$ and explain why the sequence $Y_1, Y_2, \ldots, Y_n, \ldots$ does not form a Markov chain. (3)

(ii) Draw the transition graph of the Markov chain X and write down its transition matrix. (4)

(iii) Examine (be careful to explain your reasons in each case):

a) whether the Markov chain X is time-homogeneous

b) whether it is irreducible

c) whether it admits a stationary probability distribution (4)

(iv) Starting from the state $X_t = j$, calculate the probability of suffering no further accident for the next n successive periods. (2)

(v) Suppose you are provided with full claims records for a number of a company's policy holders.

a) Describe a method for estimating the parameters X and p. (2)

b) Explain how to test the assumption that the probability of an accident depends only on the cumulative number of accidents X_n, and does not have a direct dependence on n. (2)

[17]

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