The Institute of Actuaries of India

Subject CT1 – Financial Mathematics

21st May 2007

INDICATIVE SOLUTION

Introduction

The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

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Chairperson, Examination Committee

Q.1)

(i) The effective rate of interest over a given time period is the amount of interest a single initial investment will earn at the end of the time period, expressed as a proportion of the initial amount.

(ii) $i_h(t)$, the nominal rate of interest per unit time on transactions of term h beginning at time t, is defined to be such that the effective rate of interest for the period of length h beginning at time t is $hi_h(t)$,

(iii) We want $i^{(4)}$ where i is effective rate of interest p.a.

1+i = (1.008)^12

i = 0.100339

Hence

$(1+ i^{(4)}/4)$^4 = 1.100339

$i^{(4)}$ = 4((1.100339)^(1/4)-1) = 9.677%

(iv) We want $i^{(4)}$ where i is effective rate of interest p.a.

1+i = (1.16)^(1/2) =1.077032

i = 0.077032

Hence

$(1+ i^{(4)}/4)$^4 = 1.077032

$i^{(4)}$ = 4((1.077032)^(1/4)-1) = 0.07490 ie 7.49%

$(1+ i^{(4)}/4)$^4 = 1.100034

$i^{(4)}$ = 4((1.100034)^(1/4)-1) = 9.65%

[8]

Q.2) Let A (t, t +h) be the accumulation factor from time t to time t+h and f(t, r) be the r-year forward rate from time t.

(i) Forward rate

((1+f(5,2))^2 = A(0,7)/A(0,2)A(2,5) = 1.039^7/(1.04)^2* 1.036^3

= 1.08683

Therefore

f(5,2) = 1.08683^(1/2) –1 = 0.0425

ie the 2-year forward rate from time 5 is 4.25%

(ii) Present value

The present value is given by:

=100[1/A(0,1)+1/A(0,2)+1/A(0,3)+1/A(0,4)+1/A(0,5)+1/A(0,6)+ 1/A(0,7)]

= 100[1/1.0425 + 1/1.04^2 + 1/1.0375^3 + 1/1.035^4 + 1/(1.04^2*1.036^3)+ 1/(1.0375^3*1.041^3) + 1/1.039^7]

=604.096

(iii) Accumulated value

Accumulated value is given by

100A(1,6) = 100 A(1,3) A(3,6) = 100 * (A(0,3)/A(0,1)) * A(3,6)

= 100* (1.0375^3/1.0425) * 1.041^3

= 120.848

[5]

Q.3)

(i) Proof by General Reasoning

$a^{(p)}_{\overline{n|}}$ is present value of an annuity of 1 per annum in arrears payable p times in a year for n years. That is present value of np payments, each of amount 1/p, payable at the end of each pthly interval over n year period.

We know that, by definition, a series of p payments, each of amount $i^{(p)}/p$ in arrear at pthly subinterval over a year, has the same value as single payment of amount 'i' at the end of the year.

By proportion, p payments, each of amount 1/p in arrears at pthly subintervals over a year, have the same value as a single payment of amount $i/ i^{(p)}$ at the end of each year.

Thus, np payments, each of amount 1/p, will have the same value as a series of n payments, each of amount $i/ i^{(p)}$, at time 1,2,3,……,n. This means that

$a^{(p)}_{\overline{n|}} = i/ i^{(p)}\ a_{\overline{n|}}$

(ii) Let X be the amount of withdrawal per quarter, then

$4Xa^{(4)}_{\overline{12}|} + 75000v^{12} = 500000$ @ 6% pa

$4X\, i/i^{(4)} a_{\overline{12}|} + 75000v^{12} = 500000$

$i/i^{(4)} = 1.022227$, $a_{\overline{12}|} = 8.3838$, v12= 0.49697

X = (500000-75000*0.49697)/(4*1.022227*8.3838)

= 13498.23

(iii) Present value

$PV = 100[(1+v^{1/4}+v^{2/4}+v^{3/4}) + 1.06v(1+v^{1/4}+v^{2/4}+v^{3/4}) +$ --------

$+ 1.06^{19} v^{19} (1+v^{1/4}+v^{2/4}+v^{3/4})]$ @ 6% pa

$PV = 100((1+v^{1/4}+v^{2/4}+v^{3/4})[1+1.06V+1.06^2 v^2+ \ldots + 1.06^{19} v^{19}]$ @ 6%

$= 100((1+v^{1/4}+v^{2/4}+v^{3/4})[1+1+1+\ldots+1]$

$= 2000 (1+v^{1/4}+v^{2/4}+v^{3/4})$ @ 6%

= 2000 (1+ 0.985538+ 0.971285 + 0.957239)

= 7828.12

[14]

Q.4)

(i) The n-year par yield represents the coupon per Rs.100 nominal that would be payable on a bond with term n years, which would give the bond a current price under the current term structure of Rs100 per Rs 100 nominal, assuming the bond is redeemed at par.

That is, if y_{cn} is the n-year par yield:

$1 = y_{cn}(v_{y1} + v^2_{y2} + v^3_{y3} + \text{-------} + v^n_{yn}) + 1v^n_{yn}$

(ii) The 5-year par yield y_{c5} is found from the equation:

$y_{c5}(v_{y1} + v^2_{y2} + v^3_{y3} + \text{-------} + v^5_{y5}) + 1v^5_{y5} = 1$

$y_{c5}(1.055^{-1} + 1.059^{-2} + 1.0625^{-3} + 1.065^{-4} + 1.0675^{-5}) + 1.0675^{-5} = 1$

$y_{c5} = 6.68\%$

(iii) The forward price is given by $K=S_0 e^{(\delta - D)t}$ where S_0 is the current price, D is the dividend yield, δ is the risk free force of interest and T is the length of the forward contract.

So here we have a forward price of:

$K = 1{,}000{,}000 \times e^{(0.043-0.030)\times 10} = 1138828.38$

(iv) If a new contract were to be set up in 2 year's time, the forward price would be:

$K_t = 976{,}500 \times e^{(0.043-0.030)\times 8} = 1083522.84$

So the value of the contract at this time would be:

$(1083522.84-1138828.38) \times e^{-0.043\times 8} = -39207.7$

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Q.5)

(i) Let monthly repayment be X. Working in years

$500{,}000 = 12X\, a^{(12)}_{\overline{15}|} + v^5 \times 12 \times 500\, a^{(12)}_{\overline{10}|} + v^{10} \times 12 \times 500\, a^{(12)}_{\overline{5}|}$ at 10%

$= 12X\, i/i^{(12)} \times a_{\overline{15}|} + v^5 \times 12 \times 500\, i/i^{(12)} \times a_{\overline{10}|} + v^{10} \times 12 \times 500\, i/i^{(12)} \times a_{\overline{5}|}$

At 10%,

$i^{(12)} = 12 \times \{(1.10)\text{^}(1/12) - 1\} = 0.095689$

$i/i^{(12)} = 1.045045$

$500{,}000 = 12 \times 1.045045 [X \times 7.60608 + 0.62092 \times 500 \times 6.14457 + 0.38554 \times 500 \times 3.79078]$

Hence, X = [500000/ (12 × 1.04505) - 2638.39186]/ 7.60608

= 4895.05

(ii) Interest in 1st month

$= 500000 * i^{(12)} / 12 = 3987.04$

Capital repaid = 4895.05 - 3987.04 = 908.01

(iii) O/S loan on 1 January 2007 is

$12 \times (4895.05+500) \times a^{(12)}_{\overline{8}|} + v^3 \times 12 \times 500\, a^{(12)}_{\overline{5}|}$

$12 \times 5395.05 \times i/i^{(12)} \times a_{\overline{8}|} + v^3 \times 12 \times 500 \times i/i^{(12)} \times a_{\overline{5}|}$

$12 \times 1.045045 \times (5395.05 \times 5.33493 + 0.75131 \times 500 \times 3.79078)$

378802.56

(iv) Let annual amount be Y

$Y \times a_{\overline{4}|} = 378802.56$
$Y = 378802.56 / 3.16987$
$= 119500.98$

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Q.6)

(i) (a) The investor may need to reinvest the coupon payments. The terms that will be available for reinvestment are not known at outset.

(b) For an investor who plans to sell before redemption, the ultimate sale price is not known at outset.

(c) The real return (ie in excess of inflation) is uncertain. If inflation turns out to be higher than expected at outset, the real returns from fixed interest bonds will be lower than originally anticipated.

(d) Tax rates may change, affecting the income and capital proceeds received by the investor

(ii) We will work in years, assuming a holding of £100 nominal.

The yield is the interest rate satisfying:

$90 = 7a^{(2)}_{\overline{15}|} + 105\, v^{15}$

A rough guess can be calculated as

7/90 +((105-90)/15)/90 = 8.88 approx

Using trial and error:

At 8.8%: $7a^{(2)}_{\overline{15}|} + 105\, v^{15} =$ 58.32689+29.63165= 87.95854

At 8%: $7a^{(2)}_{\overline{15}|} + 105\, v^{15} =$ 61.09162+33.10038= 94.1920

By interpolation:

(i-8)/(8.8-8) = (90– 94.1920)/(87.95854-94.1920)

i = 8 +0.8* 0.67249 = 8.538

[9]

Q.7)

(i) The other considerations are:

Cashflow
Borrowing requirements
Resources
Risk
Investment conditions
Cost vs benefit
Indirect benefits

(ii) The Loan of Rs.500000 will require interest payment of Rs.40000 at the end of each year and repayment of Rs 500000 at the end of 6 years.

The income from the project is Rs. 30000 in first year which increases by 8% compound each year.

So the cash flows in each year will be as follows:

Year	Interest due on original loan	Income from the project
1	40000	30000
2	40000	32400
3	40000	34992
4	40000	37791.36
5	40000	40814.67
6	40000	44079.84

During first 4 years, the income will not be sufficient to pay the interest on the original loan. So extra loan will be required, which will need to be repaid at the end of 5 years. The accumulated value of the debt can be calculated as:

$(40000\, a_{\overline{4}|} \;-\; 30000(v + 1.08v^2 + 1.08^2v^3 + 1.08^3v^4))(1+i)^6$ @ 8%

$(40000\, a_{\overline{4}|} \;-\; 30000/1.08(1.08v + 1.08^2v^2 + 1.08^3v^3 + 1.08^4v^4))(1+i)^6$ @ 8%

$(40000\, a_{\overline{4}|}$ @ 8% - $30000/1.08\, a_{\overline{4}|}$ @ j%) $(1.08)^6$

1+J = 1.08/1.08 =1, hence j=0

(40000*3.3121 – 30000/1.08*4) * 1.58687 = 33916

In the 5th and 6th years, there will be surplus of 814.67 and 4079.84 respectively, which will accumulate at 6% to

814.67 *1.06 + 4079.84 = 4943.39
So the overall accumulated profit/loss will be
650000-500000-33916 + 4943.39 = 121027.39

[13]

Q.8)

(i) $S_n = (1 + i_1)(1 + i_2) \ldots (1 + i_n)$
$E(i_1) = E(i_2) = \ldots = E(i_n) = j$
$Var(i_1) = Var(i_2) = Var(i_n) = s^2$
$E(S_n) = E[(1 + i_1)(1 + i_2)(1 + i_3) \ldots (1 + i_n)]$
Due to independence
$= E(1 + i_1)\, E(1 + i_2)\, E(1 + i_3) \ldots E(1 + i_n)$
$= (1 + j)^n$
$Var(S_n) = E[S_n^2] - (E[S_n])^2$
$E(S_n^2) = E[(1 + i_1)^2 (1 + i_2)^2 \ldots (1 + i_n)^2]$
Due to independence
$= E[(1 + i_1)^2]\, E[(1 + i_2)^2] \ldots E[(1 + i_n)^2]$
$= \{1 + 2j + E(i^2)\}^n$
$E(i^2) = Var(i) + j^2$
$E(i^2) = s^2 + j^2$
$E(S_n^2) = (1+2j + j^2 + s^2)^n$
$Var(S_n) = E(S_n^2) - E(S_n)^2$
$Var(S_n) = (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}$

(ii) (a) Expected value of S_{10}
$E[S_{10}] = 1.05^{10} = 1.62889$

(b) Variance of S_{10}
$Var(S_{10}) = (1 + 2 \times 0.05 + 0.05^2 + 0.07^2)^{10} - (1.05)^{20}$
$= 2.71163 - 2.65329 = 0.120311$
Standard deviation is 0.346858

[8]

Q.9)

(i) Large unit sizes, leading to less flexibility than investment in shares.
Each property is unique, so can be difficult to value. Valuation is expensive, because of the need to employ an experienced surveyor.
The actual value obtainable on sale is uncertain: property markets can crash just as stock markets can.
Buying and selling expenses are higher than for shares and bonds.
Maintenance expenses may reduce net rental income.
There may be periods when the property is unoccupied, and no income is received

(ii) The monthly rent is Rs.15000 and so the annual rent is Rs. 180000.
The purchase price
$P = 180000[adue^{(12)}_{\overline{3}|} + (1.15) \times v^3 \times adue^{(12)}_{\overline{3}|} + (1.15)^2 \times v^6 \times adue^{(12)}_{\overline{3}|}$
$\ldots\ldots + (1.15)^{11} \times v^{45} \times adue^{(12)}_{\overline{3}|}\,] + v^{48} \times (1.10)^{48} \times P$
$P = 180000 \times adue^{(12)}_{\overline{3}|}[1+(1.15) \times v^3 + (1.15)^2 \times v^6 + \text{-------} (1.15)^{11} \times v^{45}]$
$+ v^{48} \times (1.10)^{48} \times$
$0.57890P = 180000 \times adue^{(12)}_{\overline{3}|}[1 + (1.15) \times v^3 + (1.15)^2 \times v^6 + \text{-------} (1.15)^{11}$
$\times v^{45}]$ ---(a)
$adue^{(12)}_{\overline{3}|\,@\,12\%} = i/d^{(12)}\, a_{\overline{3}|} = 1.063875 \times 2.4018 = 2.55521$
$[1+(1.15) \times v^3 + (1.15)^2 \times v^6 + \text{-------} (1.15)^{11} \times v^{45}]$ @ 12%
$= adue_{\overline{16}|\,@\,j\%}$
Where $1+j = 1.12^{\wedge 3}/1.15$,
$J = 22.168\%$
Thus, $adue_{\overline{16}|\,@\,j\%} = (1 - 1.22168^{-16})/(0.22186/1.22168)$
$= 5.28718$

From equation(a)

0.57890P = 180000*2.55521*5.28718 = 2431773.937

P = 4200680

[9]

Q.10)

(i) The conditions are

(a) The value of the assets at the starting rate of interest is equal to the value of the liabilities.

(b) The volatilities (or durations) of the assets and liabilities cashflows series are equal.

(c) The convexity of the assets cashflow series is greater than the convexity of the liability cashflow series.

(ii) Let A be the amount of first liability payment at time t then the second liability payment amount will be 2A and will be payable at time (t+4)

The present value of liability

$Av^{t} + 2Av^{(t+4)} = 150000$ ---------------- (a)

DMT at 8% is 7

$$\frac{Atv^{t} + 2A(t+4)v^{(t+4)}}{Av^{t} + 2Av^{(t+4)}} = 7$$

$$\frac{t+2(t+4)v^{4}}{1+2v^{4}} = 7$$

$t(1+2v^4) + 8v^4 = 7+14v^4$

$t = (7+6v^4)/(1+2v^4) = 11.41018/2.47006 = 4.6193$

Substituting value of t in equation (a), we have

$A = 150000/((1+2v^4)\, v^{4.6193} = 150000/1.731064 = 86651.91$

Thus, the payments are 86155.91 after 4.62 years and 173303.82 after 8.62 years.

(iii) The present value of asset is equal to present value of liability payouts.

The duration of asset is equal to duration of liabilities.

The convexity of the assets is less than the convexity of the liability because asset cashflow falls between the liability cashflows.

The first two conditions are met but last condition is not met. Thus portfolio does not satisfy the Redington's immunization conditions.

[9]

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